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LEADER-FOLLOWER AND NASH STRATEGIES WITH STATE INFORMATION

GEORGE PANAYIOTOU PAPAVASSILOPOULOS

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This thesis deals with Leader-Follower and Nash differential games with state information. The main topics considered are the following: conditions for existence of closed-loop Nash strategies in linear quadratic differential games; necessary conditions for Leader-Follower differential games where the leader has current state information; sufficient conditions for Leader-Follower and Nash differential games with memory; conditions for existence, uniqueness as well as methods for finding the solution for some classes of stochastic Nash games.

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WITH STATE INFORMATION

by

George Panayiotou Papavassilopoulos

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WITH STATE INFORMATION

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THESIS

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This thesis deals with Leader-Follower and Nash differential games with state information. The main topics considered are the following: conditions for existence of closed-loop Nash strategies in linear quadratic differential games; necessary conditions for Leader-Follower differential games where the leader has current state information; sufficient conditions for Leader-Follower and Nash differential games with memory; conditions for existence, uniqueness as well as methods for finding the solution for some classes of stochastic Nash games.

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CHAPTER 1

INTRODUCTION

1.1. Motivation and General Background

The present thesis deals with certain topics in the area of Nash and Leader-Follower dynamic games with state information available to the players. Our motivation for this study was the belief that game theory provides formalisms and results which are useful in describing, understanding and manipulating successfully large scale and hierarchical engineering systems. Hierarchical and large scale systems have received considerable attention during the last few years; firstly because of their importance in engineering, economics and other areas, and secondly because of the increased capability of computer facilities [37], [38]. An important characteristic of many large scale systems is the presence of many decision makers with different and usually conflicting goals. The existence of many decision makers who interact through the system and have different goals may be an inherent property of the system under consideration (e.g., a market situation), or may be simply the result of modeling the system as such (e.x., a large system decomposed to subsystems for calculation purposes). Differential games are useful in modeling and studying dynamic systems where more than one decision maker is involved. Most of the questions posed in the area of the classical control problem may be considered in a game situation, but their resolution is generally more difficult. In addition, many questions can be posed in a game framework, which are meaningless or trivial in a classical control problem framework. The superior conceptual wealth of game over control problems, which makes them potentially much more applicable,

counterbalances the additional difficulties encountered in their solution.

The theory of games achieved its maturity as a field of active research basically due to John von Neumann. The publication of the book "Theory of Games and Economic Behavior" by J. V. Neumann and O. Morgenstern [66], gave the impetus for research in Game Theory. Although the theory of games was initially appealing almost exclusively to some Economists, it's usefulness in applications and challenge as an area of research is today recognized by many Engineers, Mathematicians, Economists, Sociologists, Psychologists and Political Scientists. The attention of the researchers was initially almost exclusively focused on static two-player, zero-sum games. Rufus Isaacs by imposing a certain type of structure to the sets and the functions describing a two-player, zero-sum game in it's abstract general form established the area of two-player, zero-sum differential games, [67]. In the literature, the characterization "static" is usually attributed to games which are presented so as to have suppressed in the final formalism any explicit dependence on the evolution of time. The characterization "dynamic" is attributed to games where evolution of time is not suppressed in the final formalism with which the game is presented. The difference between static and dynamic games lies mostly in the way that we choose to state the game, since every dynamic game can be stated equivalently as a static one. (The discussion on normal and extensive form of a game, see [66], is pertinent here). Nonetheless by adopting a static game theoritic description of a certain problem, in which problem the evolution of time is present, we hide a lot of the underlying concepts which are related to the evolution of time and might be useful in posing and studying questions about the game as well

as in interpreting several theoretical results. Dynamic games are usually
described by differential of difference equations (deterministic or stochastic)
and the term differential games is usually reserved for the former description.
The present thesis deals with dynamic games with more emphasis on the differen-
tial ones.

The two particular types of games which are of interest here are
the so called Nash differential games and Leader-Follower (LF) differential
games, see [1] and [33] respectively. The general definitions of Nash and
LF games are as follows. Let U, V be two sets and J_1, J_2 two functions

$$J_i: \quad UxV \to R, \quad i = 1, 2. \tag{1.1}$$

J_1, J_2 are referred to as the costs and U, V as the strategy spaces of
players 1 and 2 respectively.

Definition of a Nash Equilibrium: A pair $(u^*, v^*) \in U \times V$ is called a Nash
equilibrium pair if (u^*, v^*) satisfies

$$J_1(u^*, v^*) \le J_1(u, v^*), \qquad \forall u \in U$$
$$\tag{1.2}$$
$$J_2(u^*, v^*) \le J_2(u^*, v), \qquad \forall v \in V.$$

To define an LF equilibrium pair we need first to define a mapping T.
Consider the set valued mapping T

$$T: \quad U \to V, \quad u \mapsto Tu \subseteq V \tag{1.3}$$

defined by

$$Tu = \{v \mid v = \arg \quad \inf[J_2(u, \bar{v}); \ \bar{v} \in V]\}. \tag{1.4}$$

Clearly $Tu = \varphi$ if the inf is not achieved. We also consider the minimization problem

$$\inf J_1(u,v)$$

(1.5)

$$\text{subject to: } u \in U, \quad v \in Tu,$$

where we use the usual convention $J_1(u,v) = +\infty$ if $v \in Tu = \varphi$.

<u>Definition of an LF equilibrium</u>: A pair $(u^*, v^*) \in U \times V$ is called a Stackelberg equilibrium pair if (u^*, v^*) solves (1). (In LF games it is standard to say that a leader chooses $u \in U$ and has cost J_1 and a follower chooses $v \in V$ and has cost J_2).

If we are interested in a Nash equilibrium we say that we have a Nash game and if we are interested in an LF equilibrium we say that we have an LF game. Notice that we do not ask that $J_1 + J_2 = 0$, i.e. we are dealing with nonzero-sum games.

Nash games provide a formalism for describing situations of conflict where player 1 chooses $u \in U$, player 2 chooses $v \in V$, player 1 is interested in minimizing his cost J_1, player 2 is interested in minimizing his cost J_2 and the two players do not trust each other and do not cooperate. Nonetheless, each one assumes for the other that he will act in a rational way and try to satisfy (1.2). LF games provide a formalism for describing the following situation. The follower tries to minimize his cost J_2 for a given choice of $u \in U$ by the leader. The leader whose interest is to minimize J_1, knowing the follower's rational and having the privilege to choose his strategy first, wishes to announce a $u^* \in U$, such that together with the follower's reaction v^* to u^*, will result to the minimum possible J_1. Notice that in LF games the

leader chooses and declares his u first i.e. there is a hierarchy in the decision making in contrast with Nash games where the players choose and declare their choices simultaneously. Notice also that in a Nash game both players must know U,V,J_1,J_2 while in an LF game the leader must know U,V,J_1, J_2 and the follower must know $u*,V$ and J_2.

Nash games were first introduced and studied in a static framework by J. Nash [68]. The dynamic version of Nash games was first introduced and studied by Starr and Ho [1]. LF games was first introduced by von Stackelberg [26], who studied a simple static game in a _inite dimensional framework. The dynamic version of LF games was first introduced and studied by Chen and Cruz and Simaan and Cruz, [27-29]. The introduction of Nash and LF dynamic games was motivated significantly by R. Isaacs work [67] on two-person, zero-sum dynamic games.

In order to describe Nash and LF two-player differential games we consider the state equation

$$\dot{x}(t) = f(x(t),\overline{u}(t),\overline{v}(t),t), \quad x(t_o) = x_o, \quad t\in[t_o,t_f] \quad (1.6)$$

and the cost functionals

$$J_i(u,v) = g_i(x(t_f)) + \int_{t_o}^{t_f} L_i(x(t),\overline{u}(t),\overline{v}(t),t)dt, \quad i=1,2, \quad (1.7)$$

where $x(t)\in R^n$, $\overline{u}(t)\in R^{m_1}$, $\overline{v}(t)\in R^{m_2}$ and f, L_i, g_i are functions with appropriate domains, and ranges in R^n, R^1, R^1 respectively and satisfy certain continuity and differentiability assumptions. Also $u\in U$, $v\in V$ where U and V are defined below and $\overline{u}(t)$, $\overline{v}(t)$ are the values of u and v respectively, at time t. At each instant of time t player 1 has a certain information about the previous values of the trajectory and the previous values of his opponent's control

values, i.e. about x_t and $\bar{v}_t$. More precisely, at time t, player 1 knows $I_1(x_t,\bar{v}_t,t)$ see (1.9). Before player 1 chooses his u, he only knows the function I_1. Similarly player 2 knows a function I_2 which means that at time t we will know $I_2(x_t,\bar{u}_t,t)$. At the beginning of the game the players choose functions u and v and substitute $u(I_1(x_t,\bar{v}_t,t))$, $v(I_2(x_t,\bar{u}_t,t))$, in the place of $\bar{u}(t)$, $\bar{v}(t)$ in the state equation and the cost functionals. The differential equation

$$\dot{x}(t) = f(x(t),\ u(I_1(x_t,\bar{v}_t,t),v(I_2(x_t,\bar{u}_t,t),t)$$

$$x(t_o) = x_o,\ t\in[T_o,t_f] \tag{1.8}$$

is solved where

$$x_t = \{x(\tau)\,|\,t_o \leqslant \tau \leqslant t\}$$

$$\bar{u}_t = \{u(I_1(x_\tau,\bar{v}_\tau,\tau),\tau)\,|\,t_o \leqslant \tau \leqslant t\} \tag{1.9}$$

$$\bar{v}_t = \{v(I_2(x_\tau,\bar{u}_\tau,\tau),\tau)\,|\,t_o \leqslant \tau \leqslant t\}$$

and the u,v are fixed. Assuming that the solution of (1.8) exists and is unique, call it x(t;u,v) we can calculate the values of the cost functionals which will depend on u and v. The functions I_1,I_2, their images and the possible continuity differentiabily and image assumptions on u and v determine U and V. For example, if $I_i(\alpha,\beta,\gamma) = (\alpha(t_o),\gamma)$, i=1,2, i.e. both players have no information except knowledge of the current instant of time t and x_o we can consider U, V to be sets of piecewise continuous function of time which depend also on x_o. This case is called open-loop. Another example is to choose I_1, I_2 such that the players know at each instant of time t, the triple $\{x_o,x(t),t\}$. Notice that in general, I_1 and I_2 have function spaces

as domains and images. For the set up of this paragraph the definitions of

Nash and LF games are those given before.

In the case where (1.6) is substituted by a stochastic differential

equation and expectation is taken of the cost functionals (1.7), we say that

we have a stochastic differential game. Also, we can substitute (1.6) by a

difference (stochastic or not) equation, and consider summations (with

expectation in front), in place of the integral cost functionals (1.7).

We can pose several questions about the differential game described

above. For example, for given I_1, I_2, we can study existence and uniqueness

of solutions and necessary and sufficient conditions for a pair (u^*, v^*) to

be an equilibrium (Nash or LF) pair. We can also investigate the dependence

of an equilibrium pair and of the resulting costs on I_1 and I_2. These and

many other questions which have been posed elsewhere are difficult to study.

The case where $I_i(\alpha, \dot{\beta}, \gamma) = (\alpha(t_o), \gamma)$, i=1,2 is easier and there are several

results about it. In the present thesis we will try to answer questions on

existence, uniqueness and characterization by necessary and sufficient condi-

tions for cases where I_1, I_2 are such that provide information about the

current and the previous values of the state to the players.

1.2. Outline of the Thesis

In the rest of the introduction we give a general outline of the

results of the present work and relate them to already existing ones.

In Chapter 2 we deal with Nash two-player differential games,

where (1.6) is linear in x, $\bar{u}$, $\bar{v}$ and the costs (1.7) are quadratic, i.e.

with linear quadratic Nash games, except in the Section 2.4 where nonlinear

f and g_1, g_2, L_1, L_2 are also considered. The information available to each player at time t is x(t) and t and we call this case closed loop. The Nash solution for linear quadratic games has been studied in several papers, see [1-10]. Despite the many results available in this area, those concerning existence and uniqueness of optimal strategies are far from being satisfactory. This holds true especially if the strategies take into account information about the present and past values of the state of the system. In this context [6], [17], [18] can be pointed out. In these papers, the non-uniqueness of the Nash equilibrium strategies was demonstrated when the current state x(t) and the initial state x_o are available to at least one of the players. It was also shown that in the case of discrete-time linear quadratic games, under invertibility conditions of certain matrices, if noise is introduced to the state equation, then the Nash equilibrium strategies linear in the current state x(t) (assuming they exist) are the unique solution without restricting a priori the admissible strategies to be linear in the current state. The closed loop Nash strategies are not necessarily linear [6], and even if restriction to linear strategies is made, still little is known concerning their existence, properties, interpretation in terms of solutions to the coupled Riccati equations, and the stability of the closed loop system. For the linear quadratic game over a finite period of time [0,T], there are certain existence results for closed loop Nash strategies, assuming that T is sufficiently small and/or that the strategies lie in compact subsets of the admissible strategy spaces [3], [7], [8]. In [2], [5] the boundedness of the solutions of certain Riccati type differential equations is assumed in order to guarantee the existence of Nash strategies. Finally, [15] deals with the static N-person Nash game, under

compactness and convexity assumptions for the strategy spaces and concavity

assumptions for the criteria. Chapter 2 contains three sections. In Sec-

tion 2.2 we deal with existence of Nash equilibria which are linear in $x(t)$,

the time interval is $[0,\infty)$ and all the matrices involved in the description

of the state equation and cost functionals are constant. For this case there

was no existence result available known to us. Although our results do not

solve the problem completely they are applicable to a subclass of problems.

They are stated in terms of conditions on the norms of the matrices involved

and they do not depend on controllability or observability assumptions. They

can be viewed as conditions for solution of certain coupled algebraic Riccati

type matrix equations. We derive our basic existence result by using Brouwer's

Fixed Point Theorem in a way which disengages our result from a local character

that a straightforward application of Brouwer's Theorem would impose. The

generalization of our results to the N-player case is obvious. It should be

pointed out that for many of the conditions presented no assumptions of con-

trollability, observability or semidefiniteness are made. Therefore we

actually single out a region of parameter space (A, B_i, Q_i, R_{ij}) where in the

existence of solutions does not depend on controllability and observability.

This region is necessarily contained in the region where A is asymptotically

stable, or is the neighborhood of a parameter point for which a solution of

an auxiliary control problem exists. Outside this region the existence of

solutions will depend in general on controllability and observability proper-

ties, but presently conditions under which existence can be guaranteed are

not known. Finally as a byproduct of the use of Brouwer's Theorem we in-

terpret some existing results about the algebraic Riccati equation of the

control problem in a new way, see Remark iii in 2.1.3. In Section 2.2 we

consider a Linear Quadratic Nash Game over a finite period of time $[0,T]$.
The matrices involved are piecewise continuous functions of time, The
existence of linear closed loop Nash strategies depends on the existence of
continuous solutions to an associated system of two coupled Riccati differen-
tial equations over $[0,T]$. Sufficient conditions for existence are
derived by using a simple result from the theory of differential
inequalities. The conditions are given in terms of upper bounds on the
length of the time interval of interest and do not depend on controllability
or observability assumptions. The positive (semi-) definiteness assumptions
on Q_j, R_{ij} are not used in proving the existence of solutions.
Although the conditions give only a partial answer to the question
of existence of solutions, they can nonetheless provide a positive answer
for a certain class of problems. The extension of the present results to the
N-players case is straightforward,

In Section 2.3 we consider the case where f, L_1, L_2, g_1, g_2 are
analytic functions, the time interval is finite and fixed and the players have
closed loop information. It is shown that if the strategy spaces are re-
stricted to analytic functions of the state and time then the Nash equilibrium
pair is unique - if it exists, In particular, for a linear quadratic game,
where the matrices involved are analytic functions of time, it is shown that
if the coupled Riccati differential equations have a solution, then the Nash
equilibrium strategies which are affine functions of the state constitute the
unique analytic solution pair. Although the result of this section is proven
under the strong analyticity assumptions, it provides at least a partial

answer to the question of uniqueness for a certain class of problem. It provides also an additional characterization of Nash equilibrium strategies which are affine functions of the state in the context of linear quadratic games with analytic matrices, since it shows that these strategies constitute the solution over strategy spaces much larger than those which are apriori restricted to be affine in the state strategies. The introduction of analyticity assumptions removes the nonuniqueness of the Nash solution for deterministic differential games, which is analogous to the removal of nonuniqueness of Nash solutions by introduction of noise [17], [18]. The extension of these results to the N-player case is straightforward.

In Chapter 3 we study LF differential games where the leader knows at each instant of time t, the values x_o and x(t) of the state and of course t. In the area of LF games, the type of strategy spaces U and V which were considered and treated successfully in the previous literature where the spaces of piecewise continuous functions of time. In this case, the problem of deriving necessary conditions for the Stackelberg differential game with fixed time interval and initial condition x_o, falls within the area of classical control. Thus variational techniques can be used in a straightforward manner. The case where the strategy spaces are spaces of functions whose values at instant t depend on the current state x(t) and time t, i.e., $\bar{u}(t) = u_{|t} = u(x(t),t)$, $\bar{v}(t) = v_{|t} = v(x(t),t)$, was not treated. This case results in a nonstardard control problem because $\frac{\partial u}{\partial x}$ appears in the follower's necessary conditions. Since the follower's necessary conditions are seen as state differential equations by the leader, the presence of $\frac{\partial u}{\partial x}$ in them makes the leader face a nonstandard control problem. In this Chapter, the nonstandard

control problem arising from the consideration of the above strategy spaces

is embedded into a more general class of nonstandard control problems. The

characteristics of this general class of problems are the following:

(i) each of the components u^i, of the control m-vector u, depends on the

current time t and on a given function of the current state and time, i.e.

$u^i|_t = u^i(h^i(x(t)),t)$; (ii) the state equation and the cost functional de-

pend on the first order partial derivative of u with respect to the state

x. The vector valued functions h^i may represent outputs or measurements

available to the i-th "subcontroller," in a decentralized control setting.

The only restriction to be imposed on h^i is to be twice continuously

differentiable with respect to x. This allows for a quite large class of

h^i's which can model output feedback or open loop control laws. It can also

model mixed cases of open loop and output feedback control laws where during

only certain intervals of time an output is available. The appearance of

the partial derivative of u with respect to x prohibits the restriction of

the admissible controls to those which are functions of time only. Two

different approaches for deriving necessary conditions for the nonstandard

control problem are presented. The first uses variational techniques, while

the second reduces the nonclassical problem to a classical one. The

nonexistence of a control law $u*(x(t),t)$ which u* solves the problem for

every x_o is shown. The nonuniqueness of the solution of this problem is also

considered and explained. The results obtained for this nonstandard control

problem are used to study an LF differential game where the players have

current state information $(x_o,x(t),t))$. Necessary conditions that the opti-

mal strategies must satisfy are derived. The inapplicability of dynamic

programming to LF dynamic games is explained and discussed. The singular

character of the leader's problem is proven and the nonuniqueness of his

strategies is proven and characterized. In particular, it is shown that

commitment of the leader to an affine in the current state, time varying

strategy does not induce any change to the optimal costs and trajectory.

The Linear Quadratic LF game is also worked out as a specific application.

We will now outline certain generalizations of the results of this Chapter.

We consider, first, the discrete time versions. Consider the dynamic system

$$x_{k+1} = f(x_k, u^1(h^1(x_k,k),k), \ldots, u^m(h^m(x_k,k),k),$$

$$u_x^1(h^1(x_k,k),k), \ldots, u_x^m(h^m(x_k,k),k),k)$$

$$x_o \text{ given, } k = 0, \ldots, N-1$$

and the cost

$$J(u) = g(x_N) + \sum_{k=o}^{N-1} L(x_k, u^1(h^1(x_k,k),k), \ldots, u^m(h^m(x_k,k),k),$$

$$u_x^1(h^1(x_k,k)k), \ldots, u_x^m(h^m(x_k,k),k).$$

The proof of the corresponding Theorem 3.2.1 is straightforward. An immediate

consequence is that the restriction

$$u^i(h^i(x_k,k),k) = A_k^i h^i(x_k,k) + B_k^i, \; i=1, \ldots, m_1$$

where A_k^i, B_k^i are matrices, does not induce any loss of generality as far as

the optimal cost and trajectory are concerned. Proposition 3.2.1

carries over, too. A discrete time version of the LF game of 3-1 can be

defined (see [33]), and analyzed as in 3-3. Several information patterns can

be exploited by employing different h^i's (see [33]). The restriction of the

leader to affine strategies can also be imposed in the discrete case. The

case where higher order partial derivatives of u w.r. to x appear can be

treated, and all the analysis carries over. This case is of interest in

hierarchical systems since it arises, for example, in an N-level LF game where

the players use control values dependent on the current state and time.

Finally, an N-level LF game where on each i-level (i=1,...,N), n_i followers

operate $(u_1^i,...,u_{n_i}^i)$, play Nash (or Pareto) among them, and $u_j^i| =$

$u_j^i(h_j^i(x,t),t)$, j=1,...,n_i, i=1,...,N, with given h_j^i and fixed x_o, t_o, t_f can

be easily treated by using the analysis for the nonstandard control problem

supplied in 3-2.

In Chapter 4, we consider a continuous time two player deterministic

differential game with a linear state equation and two quadratic cost functionals.

We consider the case where the players have at each instant of time recall of

previous values of the trajectory, i.e. they have memory. What they remember

about the previous values of the trajectory, is allowed to change with the

elapse of time. It is a known fact that in LF and Nash differential games, the re-

sulting trajectory and strategy values vary with the admissible strategy spaces.

Most of the results available until now deal with cases where the current state

or the initial state of both of them are the only available information to the

players. A more general situation is to assume that at each instant of time,

each player knows something about the previous values of the state of the

system and about the previous values of his and the other player's decisions.

The first attempt to derive necessary conditions for zero sum games where the

strategies depend at each instant of time t on the part of the state trajectory

between t-r and t, where r>0, appears to be in [40]. In [41], [42] the zero-sum case is considered where one player has a time lag information on the value of the state. In [42] a Hamilton-Jacobi theory is developed for such games. In Chapter 3 the LF differential game was solved when the leader's information at time t is $x(t)$, $x(0)$, and t. It was shown there, that the leader can in general restrict himself in strategies affine in $x(t)$ and that use of nonlinear strategies in $x(t)$ will not improve his cost. The argument of Chapter 3 can in principle be extended to the case where the leader's information at time t is $\{x(\theta), t_0 \leq \theta \leq t\}$ and one can show that the leader does not in general deteriorate his cost if he uses strategies affine in $\{x(\theta), t_0 \leq \theta \leq t\}$. Therefore one is motivated to restrict apriori the strategy of the leader to be of the form $\int_t^t [d_s \eta(t,s)]x(s) + b(t)$, in which case η and b are what the leader will actually choose. For given η and b the follower solves his problem. Necessary and sufficient conditions for the follower's control problem can be found in [45] and [46] (Theorem 5.2) respectively. On the other hand, the leader's problem is quite difficult since his unknowns are η and b. It was also shown in Chapter 3 that the principle of optimality holds in LF games if and only if the leader's problem can be treated as a team problem for both leader and follower. (This does not necessarily mean $J_1=J_2$). These remarks motivate us to study LF games where the solutions are linear in $\{x(\theta), t_0 \leq \theta \leq t\}$ and constitute a team solution for the leader's problem. In our model a wide range of delayed information structures is included, from perfect recall of the previous trajectory to recall of only one previous value of the trajectory. Cases where information about the past

strategy values is available to the players are also considered. We consider
strategies affine in the available information and represent them by using
Lebesgue-Stieltjes integrals. Both Nash and LF equilibrium concepts are
considered and sufficient conditions are developed for a particular but
quite interesting class of problems. Particular emphasis is placed on the
LF case. The problem that we deal with differs from those considered by
Halanay in [40] and by Ciletti in [41], [42]. Halanay considers the zero
sum case only and he allows the strategy values at time t to depend on the
part of the trajectory between $t-\tau$ and t, where $\tau>0$ is fixed. Ciletti
considers also the zero sum case and allows dependence of the strategy
values at time t only on $x(t-\sigma)$ and the strategy values between $t-\sigma$ and t,
where $\sigma>0$ is fixed. Existence and uniqueness conditions related to the
sufficiency conditions proved in this Chapter are not as yet known. Our
results generalize trivially to the N player case for a Nash game and to
the one leader-N followers case for an LF game. Although for the time being
our results are not accompanied by computationally efficient procedures,
they are of importance since they provide valid characterizations.

The last Chapter 5 deals with Nash games in the presence of
stochastic disturbances. It is known that the study of stochastic games is
usually more difficult than the study of deterministic ones and consequently
very few interesting and general results are known. The aim of this chapter
is to solve some special classes of Nash games and to point out several difficulties
which make the explicit solution of such games very difficult. It is
shown that in linear quadratic Nash games, decoupling of the information
of each player from the influence of the control of the other is not always
sufficient to establish existence and uniqueness of solutions affine in the

information. Conditions where this decoupling property sufficies to achive

that are exhibited for both static and dynamic problems. It is also shown

that decoupling of the information of each player from the control of the

other and decoupling the cost functional of the one player only from the control

of the other is very close to being sufficient for existence and uniqueness of affine

in the information solutions. Conditions where this is the case are exhibited.

We start in Section 5.2 by considering a static stochastic Nash game, where

each player has a quadratic cost and his information is a linear function of

a Gaussian random variable. Certain known results are considered first and

some new ones are provided concerning the existence and uniqueness of the

solution. In 5.3 we study solutions of the game of 5.2 which are affine in the

information and provide a method for finding them. In 5.4 we generalize some

of the results of the previous sections to the case where each player's control

vector is subdivided to smaller control vectors each one of which has to use

different information. The information available to the subvector of each

control vector is nested. In 5.5 we consider a discrete time stochastic Nash

game with linear stochastic state equation and quadratic costs where the

players have noise corrupted state measurements. Special cases of this game

were solved in [17] and [63]. The case where both players have perfect state

measurements. was studied in [17] and it was shown that if the noise in the

state equation is nondegenerate, then the game admits a unique solution affine

in the information, under invertibility conditions for certain matrices. It

was also shown in [17] that if the noise in the state equation is degenerate,

then the game will have in general an infinite number of nonlinear solutions.

The case where at each stage k the players share their previous state

measurements and their information differ only in the k-th

state measurement was studied in [63] where it was shown that the

game will admit a unique solution affine in the information, under

invertibility conditions for certain matrices. In the more general case

where the assumptions of [17], [63] concerning the information of the players

do not hold, the solutions of the game becomes extremely difficult. In Section

5.5 we single out some new classes of problems which can be relatively easily

solved by using the results of the previous Sections. In the last section

We translate some of the results of Section 5.5 to a continuous time

stochastic Nash game with a linear stochastic state equation and quadratic

costs where the players noise corrupted state measurements. Examples

demonstrating certain properties of the solutions are considered in most

Sections.

Finally we have a Conclusions Chapter 6 where we outline directions

for future research and seven Appendices.

1.3. Notation and Abbreviations

R^n: n-dimensional real Euclidean space with the Euclidean metric

$\| \ \|$ or $| \ |$: denotes the Euclidean norm for vectors and the sup norm for matrices

$'$: denotes transposition for vectors and matrices.

For a function $f: R^n \to R^m$ we say that $f \in C^k$ if f has continuous

mixed partial derivatives of order k. For $f: R^n \to R$, ∇f is considered as

$n \times 1$ column vector and f_{xx} denotes the Hessian of f. For $f: R^n \to R^m$, ∇f is

considered as $n \times m$ matrix (Jacobian). For $f: R^n \times R^k \to R^m$, where $x \in R^n$, $y \in R^k$,

we denote by $\frac{\partial f}{\partial x}$ or f_x or $\nabla_x f$ the Jacobian matrix of the partial

derivatives of f with respect to x and is considered as $n \times m$ matrix.

$C([t_0, t_f], R^n) = C_n$: The Banach spaces of continuous function $\varphi: [t_0, t_f]$

$\varphi: [t_0, t_f] \to R^n$, with norm

$$\|\varphi\| = \sup\{|\varphi(t)|; \ t \in [t_0, t_f]\}$$

where $|\ |$ denotes the usual Euclidean distance in R^n.

$L_1([t_0, t_f], R^n) = L_{1,n}$: The Banach space of Lebesgue integrable functions

$\varphi: [t_0, t_f] \to R^n$ with norm $\|\varphi\| = \displaystyle\int_{t_0}^{t_f} |\varphi(t)|dt$.

$L_\infty([t_0, t_f], R^n) = L_{\infty,n}$: The Banach space of Lebesgue measurable functions which are almost everywhere bounded, with norm $\|\varphi\| = \mathrm{ess\ sup}\{|\varphi(t)|, t \in [t_0, t_f]\}$.

$NBV([t_0, t_f], R^n) = NBV$: The Banach space of normalized functions of bounded variation, i.e.: continuous from the right on (t_0, t_f), zero at t_f, and $\|\varphi\| = \mathrm{Var}(\varphi)$ for $\varphi \in NBV$.

A norm in one of these spaces is denoted sometimes by $\|\cdot\|_C$, $\|\cdot\|_{L_1}$, $\|\cdot\|_{NBV}$. B^* denotes the conjugate space of a Banach space B. If $x^* \in B^*$ and $x \in B$, we write $\langle x^*, x \rangle = x^*(x)$.

w.r. to: with respect to

w.l.o.g.: without loss of generality

n.b.d.: neighborhood.

Additional notation is introduced when needed.

CHAPTER 2

CLOSED LOOP NASH STRATEGIES

2.1. Introduction

The present Chapter deals with closed loop Nash solutions to continous
time differential games and is divided into three parts, 2.2, 2.3, and 2.4.
In 2.2 and 2.3 we are concerned with existence of solutions to linear quadratic
Nash differential games and in 2.4 with uniqueness of analytic solutions to Nash
differential games with analytic data.

In 2.2 we consider the linear quadratic, time invariant case over
an infinite horizon. The structure of 2.2 is the following. In 2.2.1 we
describe the system and formulate the problem which is the existence of
linear closed loop Nash solutions which result in a stable system. The ques-
tions posed are pursued in 2.2.2 and 2.2.3. The existence of such solutions
depends on the existence of solutions to a system of two coupled algebraic
Riccati equations which result in a stable closed loop system. Conditions
for the existence of such solutions are derived via Brouwer's Fixed Point
Theorem. The conditions derived in 2.2.2 state that linear closed loop Nash
strategies exist if the open loop matrix A has a sufficient degree of stability
which is determined in terms of the norms of the weighting matrices. 2.2.3
contains some extensions of the conditions derived in 2.2.2 which do not
require stability of the open loop matrix.

In 2.3 we consider the linear quadratic finite time ($t \in [0,T]$) case.
The matrices involved are piecewise continuous functions of time. The struc-
ture of 2.3 is the following. In 2.3.1 we describe the system and pose the
problem which is the existence of linear closed loop Nash solutions. The

existence of such solutions depends on the existence of solutions to an associated system of two coupled Riccati differential equations over $[0,T]$. In 2.3.2 sufficient conditions for existence are derived by using a simple result from the theory of differential inequalities.

In 2.4 we consider a Nash differential game where the functions f, L_i, g_i, involved in the description of the problem are analytic functions of their arguments and seek Nash solutions which are also analytic functions of their arguments. This problem is studied in 2.4.1. In 2.4.2 we apply the results of 2.4.1 to a linear quadratic, finite time, Nash game where the matrices involved are analytic functions of time.

Four Appendices to this Chapter are given at the end of the Thesis.

2.2. Infinite Horizon: Existence of Closed Loop Nash Strategies and Solutions to Coupled Algebraic Riccati Equations

2.2.1. Problem Statement

Consider the dynamic system described by

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2, \quad x(0) = x_o, \quad t \in [0,+\infty) \tag{2.1}$$

and two functionals

$$J_1(u_1, u_2) = \int_0^{+\infty} (x'Q_1 x + u_1' R_{11} u_1 + u_2' R_{12} u_2) dt$$

$$\tag{2.2}$$

$$J_2(u_1, u_2) = \int_0^{+\infty} (x'Q_2 x + u_2' R_{22} u_2 + u_1' R_{21} u_1) dt$$

where x, u_1, u_2 are functions of time taking values in R^n, R^{m_1}, R^{m_2} respectively an. A, B_1, B_2, $Q_i = Q_i'$, $R_{ij} = R_{ij}'$, $R_{ii} > 0$, $i,j = 1,2$ are real constant matrices of appropriate dimensions.

The problem is to find u_1^*, u_2^* as linear functions of x, i.e.,

$u_i^* = -L_i^* x$, with L_i^* real constant matrix, such that $J_i(u_1^*, u_2^*)$ is finite (see Appendix B), $i = 1,2$ and

$$J_1^* = J_1(u_1^*, u_2^*) \leq J_1(u_1, u_2^*) \quad \text{for every } u_1 = -L_1 x$$

$$J_2^* = J_2(u_1^*, u_2^*) \leq J_2(u_1^*, u_2) \quad \text{for every } u_2 = -L_2 x. \tag{2.3}$$

The conditions (2.3) are the Nash equilibrium conditions. It is known (see [1]), that a necessary condition for the existence of such controls u_1^*, u_2^*, is that there exist constant real symmetric matrices K_1, K_2 satisfying

$$0 = K_1 A + A' K_1 + Q_1 - K_1 B_1 R_{11}^{-1} B_1' K_1 - K_1 B_2 R_{22}^{-1} B_2' K_2 - K_2 B_2 R_{22}^{-1} B_2' K_1$$
$$+ K_2 B_2 R_{22}^{-1} R_{12} R_{22}^{-1} B_2' K_2$$

$$0 = K_2 A + A' K_2 + Q_2 - K_2 B_2 R_{22}^{-1} B_2' K_2 - K_2 B_1 R_{11}^{-1} B_1' K_1 - K_1 B_1 R_{11}^{-1} B_1' K_2 +$$
$$+ K_1 B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1' K_1 . \tag{2.4}$$

It can be proved that if such K_i's exist and the closed loop matrix

$$\tilde{A} = A - B_1 R_{11}^{-1} B_1' K_1 - B_2 R_{22}^{-1} B_2' K_2 \tag{2.5}$$

has $\operatorname{Re}\lambda(\tilde{A}) < 0$, i.e., $\tilde{A}$ is asymptotically stable (A.S.), and [2]

$$Q_i + K_j' B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j' K_j \geq 0, \quad i \neq j, \; i,j = 1,2 \tag{2.6}$$

---

[2] For (2.6) to hold it suffices for example: $Q_i \geq 0$, $R_{ij} \geq 0$.

then the strategies

$$u_i = -L_i^* x = -R_{ii}^{-1} B_i' K_i x, \quad i = 1,2 \tag{2.7}$$

satisfy (2.3) and J_1^*, J_2^* are finite. (In relation to this see Proposition 1 in [4]. The proof of this Proposition in [4] does not hold under the assumptions stated there, see Appendix A).

In the next section we will deal with the solution of (2.4) and try to find conditions under which solutions exist and yield $\tilde{A}$ A.S.

2.2.2. Conditions for Existence of Solutions

We start by introducing the following notation

$$K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}, \quad F = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}, \quad Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

$$S_1 = B_1 R_{11}^{-1} B_1', \quad S_2 = B_2 R_{22}^{-1} B_2'$$

$$S_{o1} = B_2 R_{22}^{-1} R_{12} R_{22}^{-1} B_2', \quad S_{o2} = B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1' \tag{2.8}$$

$$S = \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix}, \quad S_o = \begin{bmatrix} S_{o1} & 0 \\ 0 & S_{o2} \end{bmatrix}$$

where I denotes the $n \times n$ unit matrix. Using this notation, (2.4) assumes the form

$$0 = \Re(K) \triangleq F'K + KF + Q - KSK - KJSKJ - JKSJK + JKJS_o JKJ. \tag{2.9}$$

Consider the space X of $2n \times 2n$ real symmetric constant matrices of the form

$$Y = \begin{pmatrix} M & 0 \\ 0 & N \end{pmatrix}$$

where M and N are $n \times n$. X is a linear subspace of the space of $2n \times 2n$ real matrices. All norms of the matrices to be considered here are the sup norms ($\|A\| = \sup\{\|Ax\| : \|x\| = 1\}$), and the norms of the vectors are the square root-Euclidean norms. It is easy to see that for $Y \in X$, $\|Y\| = \max(\|M\|, \|N\|)$. We denote by I_0 the $2n \times 2n$ unit matrix and for $R \geq 0$ we set $B_R = \{Y \in X : \|Y\| \leq R\}$, i.e., B_R is the compact ball of radius R centered at the zero element of X. We define the function Φ from X into X by

$$\Phi(K) = \Re(K) + K. \tag{2.10}$$

Clearly if $K \in X$ then $\Phi(K) \in X$, and Φ is continuous. The following lemma is proved by using Brouwer's fixed point theorem (see [13]p. 161).

Lemma 2.1. If for some $R \geq 0$

$$(3\|S\| + \|S_0\|)R^2 + (\|I_0 + 2F\| - 1)R + \|Q\| \leq 0 \tag{2.11}$$

holds then there exists $K \in X$, with $\|K\| \leq R$ which satisfies $\Re(K) = 0$.

Proof. For λ a fixed real number we have

$$\Phi(K) = K(F + \lambda I_0) + (F' + (1-\lambda)I_0)K + Q - KSK - KJSKJ - JKSJK + JKJS_0JKJ$$

from which for $K \in B_R$, using the obvious fact: $\|J\| = 1$, we get

$$\|\Phi(K)\| \leq R(\|\lambda I_0 + F\| + \|(1-\lambda)I_0 + F\|) + \|Q\| + R^2(3\|S\| + \|S_0\|).$$

Since $\|I_0 + 2F\| \leq \|\lambda I_0 + F\| + \|(1-\lambda)I_0 + F\|$, with equality for $\lambda = \frac{1}{2}$, we set $\lambda = \frac{1}{2}$, (best λ). The result now follows by direct application of Brouwer's theorem.   □

Let us introduce the transformation

$$K = \alpha \overline{K} \qquad (2.12)$$

where $\alpha \neq 0$ is a constant and $\overline{K} \in X$. Substituting $K = \alpha \overline{K}$ in (2.9) we obtain

$$0 = \Re(K) = \Re(\alpha \overline{K}) \overset{\Delta}{=} \Re_\alpha(\overline{K}) = (\alpha F)'\overline{K} + \overline{K}(\alpha F) + Q - \overline{K}(\alpha^2 S)\overline{K} - \overline{K}J(\alpha^2 S)\overline{K}J$$
$$\qquad (2.13)$$
$$- J\overline{K}(\alpha^2 S)J\overline{K} + J\overline{K}J(\alpha^2 S_0)J\overline{K}J.$$

Applying Lemma 2.1 to $\mathfrak{F}_\alpha(\overline{K}) \overset{\Delta}{=} \Re_\alpha(\overline{K}) + \overline{K}$, we obtain that if for some $R \geq 0$ it holds

$$(3\|S\| + \|S_0\|)\alpha^2 R^2 + (\|I + 2\alpha A\| - 1)R + \|Q\| \leq 0, \qquad (2.14)$$

then there exists $\overline{K} \in X$, $\|\overline{K}\| \leq R$ which satisfies $\Re_\alpha(\overline{K}) = 0$. But then $K = \alpha \overline{K}$ satisfies $\Re(K) = \Re_\alpha(\overline{K}) = 0$ and $\|K\| \leq |\alpha| R$. We thus have proved

Lemma 2.2. If for some $\alpha \neq 0$, $R \geq 0$, (2.14) holds, then there exists $K \in X$, $\|K\| \leq |\alpha| R$ which satisfies $\Re(K) = 0$.

The scaling introduced in (2.12) helps to improve (2.11) and get (2.14), because in proving Lemma 2.2 we applied Lemma 2.1 to a whole class of $\mathfrak{F}_\alpha$'s which are nonlinear (quadratic in $\overline{K}$) and asked that at least one of them have a fixed point via Brouwer's theorem. As it turned out if one of them, say $\mathfrak{F}_{\overline{\alpha}}$ has a fixed point then all of them have (since $\Re_\alpha(S) = \Re(\alpha S) = \Re_\beta(\frac{\alpha}{\beta}S)$) although (2.14) may not hold for $\beta \neq \overline{a}$.

Set

$$a = 3\|S\| + \|S_0\|$$

$$b = \|I + 2\alpha A\|$$
$$\qquad (2.15)$$

$$q = \|Q\|$$

$$\epsilon = \sqrt{qa} .$$

Then, (2.14) assumes the form

$$a\alpha^2 R^2 + (b-1)R + q \lesseqgtr 0 \tag{2.16}$$

If $a = 0$ then $B_1 = 0$ and $B_2 = 0$, and the game is meaningless as such. Therefore assume $a \neq 0$. Inequality (2.14) is satisfied for some $R \geq 0$ if and only if

(i) $1 \geq b + 2|\alpha|\epsilon$

or

(ii) $q = 0$ and $1 < b$.

In case (ii) $R = 0$ is the only solution to (2.16) and thus Lemma 2.2 guarantees only the solution $K_1 = 0$, $K_2 = 0$. Consequently we will concentrate on case (i), i.e., when

$$1 \geq \|I + 2\alpha A\| + 2|\alpha|\epsilon. \tag{2.17}$$

If (2.17) holds then (2.16) is satisfied for all R: $R_1 \leq R \leq R_2$ where

$$R_{1,2} = \frac{1-b \pm \sqrt{(b-1)^2 - 4\alpha^2 \epsilon^2}}{2\alpha^2 a} \geq 0. \tag{2.18}$$

In this case Lemma 2.2 guarantees the existence of solutions K_1, K_2 with $\|K_1\|$, $\|K_2\| \leq |\alpha|R$. We have

Theorem 2.1. Let $a > 0$. If for some $\alpha \neq 0$, (2.17) is satisfied, then for every R: $R_1 \leq R \leq R_2$ where R_1, R_2 are as in (2.18), there exist K_1, K_2 satisfying (2.4) such that

$$\|K_i\| \leq |\alpha|R \leq |\alpha|R_2 \leq \frac{2\|A\|}{3\|S\| + \|S_0\|} = M. \tag{2.19}$$

Proof. The proof has already been given except for the right hand side of (2.11). Since $1 = \|I + 2\alpha A - 2\alpha A\| \leq \|I + 2\alpha A\| + 2|\alpha|\|A\|$ we have

$$1 - b \leq 2|\alpha|\,\|A\|$$

and so

$$|\alpha|R_2 = |\alpha|\frac{1-b+\sqrt{(1-b)^2-4\alpha^2\epsilon}}{2\alpha^2 a} \leq \frac{(1-b)+\sqrt{(1-b)^2}}{2|\alpha|a} \leq \frac{2\|A\|}{a} = M. \qquad \square$$

Notice in passing that M is independent of the magnitude of α. Before giving the next Theorem which provides us with necessary and sufficient conditions for the existence of an $\alpha \neq 0$ satisfying (2.17), we prove the following Lemma.

__Lemma 2.3.__ Let Γ be a real $n \times n$ matrix, γ and ρ real numbers $\gamma \neq 0$, and $\lambda(\Gamma) = \sigma + jw$ be any eigenvalue of Γ. Then

(i) If

$$\|I+\gamma\Gamma\| \leq \rho \qquad (2.20)$$

then

$$\left(\sigma+\frac{1}{\gamma}\right)^2 + w^2 \leq \left(\frac{\rho}{\gamma}\right)^2. \qquad (2.21)$$

(ii) If $\gamma > 0$ and $\rho = 1$, then $\sigma < 0$ or $\sigma = w = 0$ for every $\lambda(\Gamma)$.

(iii) If $\gamma < 0$ and $\rho = 1$, then $\sigma > 0$ or $\sigma = w = 0$ for every $\lambda(\Gamma)$.

(iv) Let $\Gamma = T\Lambda T^{-1}$, where Λ is the Jordan canonical form of Γ.

We set

$$\rho' = \|T\| \cdot \|T^{-1}\| \qquad (\rho' \geq 1).$$

If

$$\|I+\gamma\Lambda\| \leq \frac{\rho}{\rho'} \qquad (2.22)$$

then (2.20) holds.

(v) If Λ is diagonal and

$$\left(\sigma+\frac{1}{\gamma}\right)^2 + w^2 \leq \left(\frac{\rho}{\rho'\gamma}\right)^2$$

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holds then (2.20) holds. In particular, if Γ is symmetric then (2.28) is equivalent to $(2.21)^2$

<u>Proof.</u> (i) Let v be an eigenvector of Γ corresponding to $\sigma + jw = \lambda(\Gamma)$ and $\|v\| = 1$, w.l.o.g. Then

$$\rho \geq \|I + \gamma\Gamma\| \geq \|(I + \gamma\Gamma)v\| = |1 + \gamma(\sigma + jw)|$$

and (2.21) follows.

(ii) This follows trivially from (2.21) by noticing the (2.21) corresponds to a disk with center at $\frac{-1}{\gamma}$ and radius $\frac{1}{|\gamma|}$, which in case $\gamma > 0$ and $\rho \leq 1$, lies in the left half plane of the (σ, jw) plane.

(iii) See (ii) above.

(iv) Trivial.

(v) This follows by using (iii). If Γ is symmetric then $T' = T^{-1}$ and $\|T'\| = \|T\| = \sqrt{\lambda\max(T'T)} = 1$ and thus $\rho' = 1$. $\quad\square$

<u>Theorem 2.2.</u> Let $\lambda(A) = \sigma + jw$ be eigenvalue of A.

(i) If $\epsilon = 0$ and (2.17) is satisfied for some $\alpha \neq 0$ then for $\lambda(A)$ it holds

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[2]The assumption that Λ is diagonal in (iv) is essential. As a counterexample let

$$\Gamma = \Lambda = \begin{pmatrix} -\frac{1}{2} & 1 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad T = T^{-1} = I, \quad \rho = 1$$

in which case (2.21) is satisfied for all γ: $0 < \gamma \leq 4$ but for $x = \frac{1}{\sqrt{2}}(1,1)'$, $\|x\| = 1$: $\|(I + \gamma\Gamma)x\| = \sqrt{1 + \gamma^2/4} > 1$.

$$\sigma = \operatorname{Re}\lambda(A) < 0 \text{ or } \lambda(A) = 0 \quad \text{if } \alpha > 0$$

$$\sigma = \operatorname{Re}\lambda(A) > 0 \text{ or } (A) = 0 \quad \text{if } \alpha < 0 \tag{2.23}$$

$$(\sigma + \frac{1}{2\alpha})^2 + w^2 \le (\frac{1}{2\alpha})^2$$

$$\|K_1\|, \|K_2\| \le \frac{1 - \|I + 2\alpha A\|}{|\alpha| a} . \tag{2.24}$$

(ii) If $\epsilon \ne 0$ and $A = -\epsilon I$, then any $0 < \alpha \le \frac{1}{2\epsilon}$ satisfies (2.19) and

$$\|K_1\|, \|K_2\| \le \alpha R_2 = \frac{\epsilon}{a}. \tag{2.25}$$

(iii) If $\epsilon \ne 0$, $A \ne -\epsilon I$ and (2.17) is satisfied for some $\alpha \ne 0$, then for any $\lambda(A)$ it holds

$$\sigma < -\epsilon \text{ or } \lambda(A) = -\epsilon \quad \text{if } \alpha > 0 \tag{2.26}$$

$$\sigma > +\epsilon \text{ or } \lambda(A) = \epsilon \quad \text{if } \alpha > 0$$

and $\|K_1\|$, $\|K_2\|$ satisfy (2.11). Moreover $|\alpha| < \frac{1}{2\epsilon}$.

(iv) If $A \ne -\epsilon I$ ($A \ne \epsilon I$) is diagonalizable, $A = T\Lambda T^{-1}$ where Λ is the Jordan canonical form of A, $\rho' = \|T\| \cdot \|T^{-1}\|$ and for some $\gamma > 0$ ($\gamma < 0$) it holds

$$(\sigma + \epsilon + \frac{1}{\gamma})^2 + w^2 \le \frac{1}{\rho'\gamma^2} \;((\sigma - \epsilon + \frac{1}{\gamma})^2 + w^2 \le \frac{1}{\rho'\gamma^2}) \tag{2.27}$$

then $\alpha = \dfrac{\gamma}{2(1 + \epsilon\gamma)}$ $(\alpha = \dfrac{\gamma}{2(1 - \epsilon\gamma)})$ satisfies (2.17)

<u>Proof</u>. (i) (2.23) follows from Lemma 2.3 (i,ii,iv) and (2.14) from (2.19).

(ii) The first part is trivial. (2.25) follows as (2.24).

(iii) Let $b = \dfrac{1}{|2\alpha|}$. Then (2.17) yields

$$b - \epsilon \ge \|bI + A\| \quad \text{if } \alpha > 0$$

$$b - \epsilon \ge \|bI + A\| \quad \text{if } \alpha < 0.$$

Necessarily $b - \epsilon > 0$. Let $\gamma = \dfrac{1}{(b-\epsilon)}$, then

$$1 \geq \|I + \gamma(A + \epsilon I)\| \quad \text{if } \alpha > 0$$

$$1 \geq \|I + \gamma(-A + \epsilon I)\| \quad \text{if } \alpha < 0.$$

We set $\Gamma = \pm A + \epsilon I$ and we have

$$1 \geq \|I + \gamma\Gamma\|. \tag{2.28}$$

(2.26) follows now from Lemma 2.3 (ii)-(iii).

(iv) We bring (2.17) to the form (2.28) and apply part

(iv) of Lemma 2.3.　□

If A is symmetric, then the existence of γ satisfying (2.27) is equivalent to the existence of α satisfying (2.17). For (2.17) to hold it suffices $1 \geq \sqrt{\mathrm{tr}[(I+2\alpha A)'(I+2\alpha A)]} + 2|\alpha|\epsilon$. By using the fact $\|M\|^2 \leq \mathrm{tr}(M'M)$ it follows that the existence of an $\alpha > 0$ satisfying (2.17) is guaranteed in the following two cases (assuming A is not a scalar).

(i) $\mathrm{tr}A \leq -\epsilon$

$\mathrm{tr}A'A < \epsilon^2$

$\Delta = (n-1)\mathrm{tr}A'A + (\mathrm{tr}A)^2 + 2\epsilon\mathrm{tr}A + (2-n)\epsilon^2 \geq 0$

$\dfrac{1}{2\epsilon} \geq \dfrac{-(\epsilon+\mathrm{tr}A) - \sqrt{\Delta}}{2(\epsilon^2 - \mathrm{tr}A'A)}$

(ii) $\mathrm{tr}A'A > \epsilon^2$

$\mathrm{tr}A \leq -\epsilon$.

We shall now consider the stability of the closed loop matrix $\tilde{A}$ (2.5). Let

$$a' = 4\|S\| + \|S_o\|$$

$$\epsilon' = \sqrt{qa'} , \quad (\epsilon'^2 = \epsilon^2 + \|S\| \cdot \|Q\|)$$

$$R_2 = \frac{1-b+\sqrt{(1-b)^2-4\alpha^2\epsilon^2}}{2\alpha^2 a}$$

$$R_2' = \frac{1-b+\sqrt{(1-b)^2-4\alpha^2\epsilon'^2}}{2\alpha^2 a'}$$

(2.29)

where a, ϵ, q, b are as in (2.15).

Theorem 2.3. Let $Re[\lambda(A)] < 0$

    (i)   If for some $\alpha > 0$ it holds

$$1 \geq \|I + 2\alpha A\| + 2\alpha\epsilon$$

(2.30.1)

$$1 > \|I + 2\alpha A\|$$

(2.30.2)

$$\alpha^2\|S\|R_2^2 < \alpha^2\|S_o\|R_2^2 + \|Q\|$$

(2.30.3)

then there exist K_1, K_2, $\|K_i\| \leq \alpha R_2$, $i = 1,2$, solving (2.9) and $\tilde{A}$ (2.5) is A.S.

    (ii)   If for some $\sim > 0$ it holds

$$1 \geq \|I + 2\alpha A\| + 2\alpha\epsilon'$$

(2.31.1)

$$1 > \|I + 2\alpha A\|$$

(2.31.2)

$$\|Q\| \text{ or } \|S_o\| \neq 0$$

(2.31.3)

then there exist K_1, K_2, $\|K_i\| \leq \alpha R_2'$, $i = 1,2$, solving (2.9) and $\tilde{A}$ (2.5) is A.S. .

Proof. (i). (2.30.1) makes Theorem 2.1 applicable. We have

$$\|I + 2\alpha\tilde{A}\| = \|I + 2\alpha(A-S_1\alpha\bar{K}_1-S_2\alpha\bar{K}_2)\| \leq \|I + 2\alpha A\| + 4\alpha^2\|S\|\|\bar{K}\|.$$

Since by Lemma 2.3 A will be A.S. if $\|I + 2\alpha A\| < 1$ for some $\alpha > 0$, it suffices

$$\|I + 2\alpha A\| + 4\alpha^2 \|S\| \|\overline{K}\| < 1.$$

(2.30.2) implies $R_2 > 0$. Using $K = \alpha \overline{K}$ and (2.19) we have that it suffices

$$(\|I + 2\alpha A\| - 1)R_2 + 4\alpha^2 \|S\| R_2^2 < 0$$

and since (2.14) holds for $R = R_2$ it suffices

$$4\alpha^2 \|S\| R_2^2 < (3\|S\| + \|S_o\|)\alpha^2 R_2^2 + \|Q\|$$

which is equivalent to (2.31.3)

(ii): (2.31.3) implies that the inequality

$$a'\alpha^2 R^2 + (\|I + 2\alpha A\| - 1)R + q \leq 0$$

is satisfied for all R: $R_1' \leq R \leq R_2'$ where

$$R_{1,2}' = \frac{1 - \|I + 2\alpha A\| \pm \sqrt{(1 - \|I + 2\alpha A\|)^2 - 4\alpha^2 e'^2}}{2\alpha^2 a'}$$

$R_2' \geq R_1' \geq 0$. (2.31.2) implies $R_2' > 0$. If the above inequality holds for some R and α, then (2.14) holds for the same R and α since $a \leq a'$. Therefore there exist K_1, K_2, $\|K_i\| \leq \alpha R_2'$, $i = 1,2$ solving (2.9). Repeating the analysis of (i) we have that for $\tilde{A}$ to be A.S. it suffices

$$4\alpha^2 \|S\| R_2'^2 < (4\|S\| + \|S_o\|)\alpha^2 R_2'^2 + \|Q\|$$

which holds by (2.31.3). □

If equality is allowed in (2.30.3) or $\|Q\| = \|S_o\| = 0$ in (2.31.3) then the conclusions of (i) and (ii) in the given Theorem change and allow $\tilde{A}$ stable, i.e., Re $[\lambda(\tilde{A})] \leq 0$.

The geometric interpretation of the conditions given in Theorems 2.2 and 2.3 is given in Figs. 1 and 2. Figure 1 corresponds to Theorem 2.2, parts i, ii, iii, which say that a necessary condition for the existence of an $\alpha > 0$ satisfying (2.17) is that the eigenvalues of A lie in a disk centered at $\frac{-1}{2\alpha}$ with radius $r = \frac{1}{2\alpha} - \epsilon$, for some $\alpha > 0$, which is equivalent to saying that all $\lambda(A)$'s lie in the open half plane on the left of the line ϵ_1, $(\sigma = -\epsilon)$, or at $-\epsilon$. Figure 2 corresponds to Theorem 2.2, part (iv), where it is assumed that A is diagonalizable. It shows that if the eigenvalues of A lie in a disk as in Figure 2 with radius $r = \frac{1}{\sqrt{\rho'}}(\frac{1}{2\alpha} - \epsilon)$ and centered at $\frac{-1}{2\alpha}$ then this α satisfies (2.17) and thus (2.9) has a solution. If A is symmetric then $p' = 1$ and $\theta = 90°$. If $\alpha < 0$, then we have the mirror images with respect to the jw axis of the circles, cones, lines depicted in Figures 1, 2.

Employment of a different function Φ in (2.10) and application of Brouwer's theorem may in general provide different, perhaps better, existence results. Another suitable Φ can be defined as follows. If K solves (2.9) and A (and thus F) is A.S., then $\overline{K} = \frac{K}{\alpha}$, where $\alpha > 0$, solves equivalently (see Appendix C)

$$\overline{K} = \int_0^{+\infty} e^{\alpha F t}[Q - \overline{K}\alpha S\overline{K} - \overline{K}J\alpha^2 S\overline{K}J - J\overline{K}\alpha^2 SJ\overline{K} + J\overline{K}J\alpha^2 S_0 J\overline{K}J] \cdot e^{\alpha F' t} dt. \qquad (2.32)$$

Let $\tilde{\Phi}_\alpha(\overline{K})$ denote the right hand side of (2.32). Let also $A = T\Lambda T^{-1}$, Λ being the Jordan canonical form of A with $m \times m$ the dimension of the largest Jordan block $(m \leq n)$. Let also a, q, ϵ be as in (2.15), $\alpha > 0$, and

$$\rho = \|T\|\|T^{-1}\|$$

$$\overline{\lambda} = \max \text{ Re } [\lambda(A)] < 0 \qquad (2.33)$$

$$\pi(w) = \sum_{i,j=0}^{m-1} \frac{(i+j)!}{i! \, j!} (\frac{w}{2})^{i+j+1} \quad \text{for } w > 0,$$

$$\overline{\pi} = \pi(1/(-\alpha\overline{\lambda}))$$
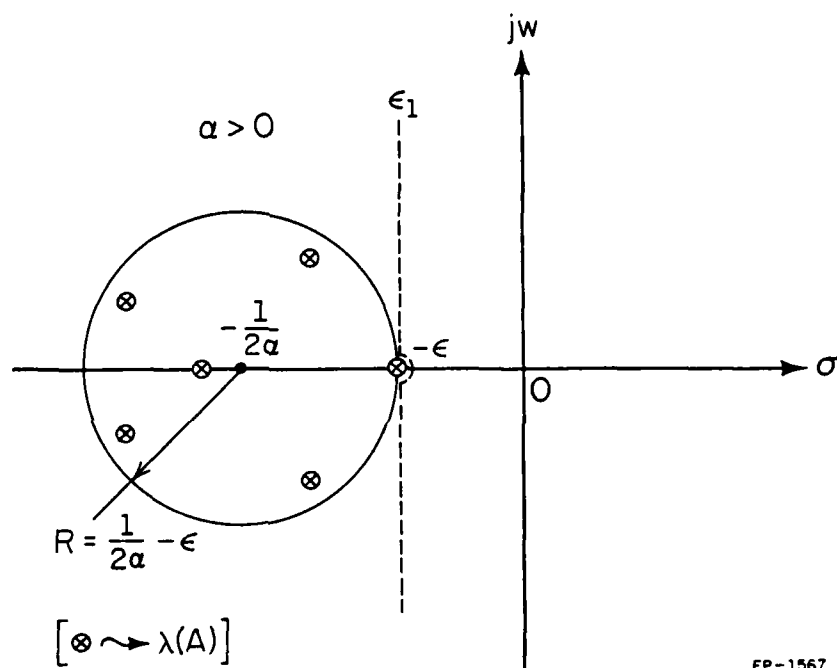
Figure 1. Regions of eigenvalues of A in accordance to Theorem 3.2 (i)-(iii).
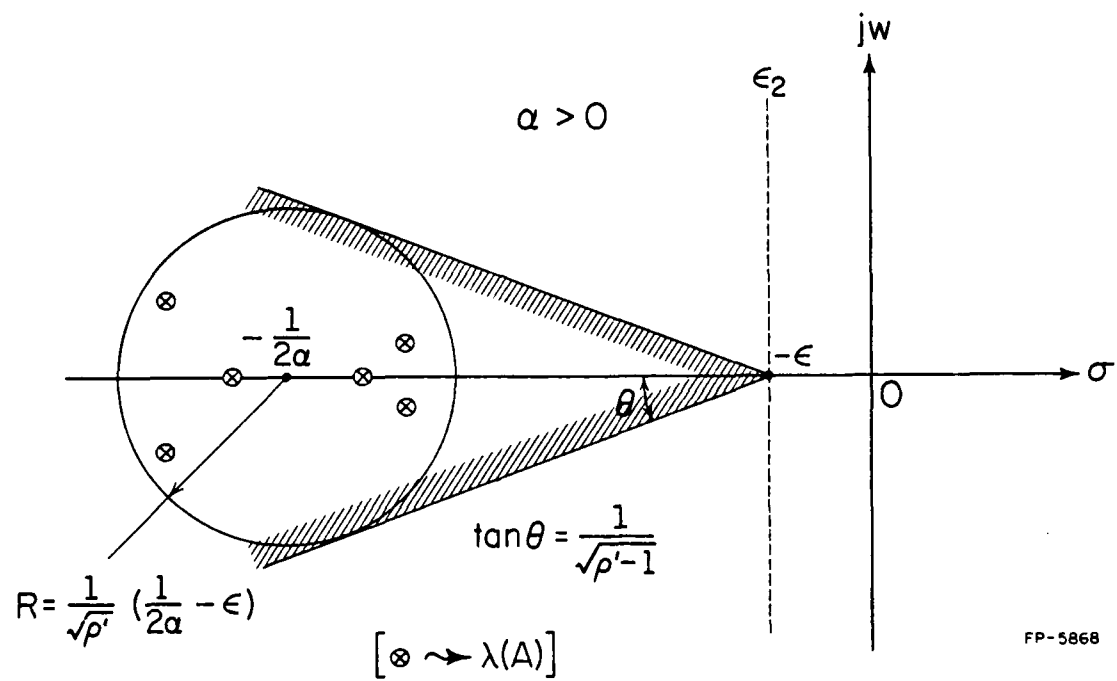
Figure 2.   Regions of eigenvalues of A in accordance to
Theorem 3.2 (iv).

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and $\epsilon(m,\alpha) > 0$ be such that

$$\pi(1/\epsilon(m,\alpha)) = \frac{1}{2\alpha\epsilon} \ .$$

Clearly $\epsilon(m,\alpha)$ exists and is unique, for given α and m.

Theorem 2.4. Let A be A.S. and T, Λ, m, π, $\epsilon(m,\alpha)$ as above. If it holds that

$$\bar{\lambda} \leq -\epsilon(m,\alpha) \cdot \frac{1}{\alpha} \tag{2.34}$$

then there is $K \in X$ which satisfies (2.9), and

$$\|K\| \leq \alpha R_2 = \alpha \ \frac{1}{2a\alpha^2} \ \{ \frac{1}{\rho^2 \cdot \bar{\pi}} + \sqrt{\frac{1}{\rho 4\bar{\pi}^2} - 4qa\alpha^2} \ \} \leq \frac{2|\bar{\lambda}|}{a\rho^2} \ . \tag{2.35}$$

In addition, if A is diagonalizable (i.e., $m = 1$) then $\epsilon(1,\alpha) = \alpha\epsilon\rho^2$ for every $\alpha > 0$.

Proof. Let $\bar{K} = \alpha K$, $\alpha > 0$, $\|\bar{K}\| \leq R, R \geq 0$. In order to use Brouwer's theorem we ask for $\|\Phi_\alpha(K)\| \leq R$ for some α and R. It suffices

$$\|\Phi_\alpha(\bar{K})\| = \| \int_o^{+\infty} e^{\alpha Ft} [Q - \bar{K}\alpha^2 S\bar{K} - \bar{K}J\alpha^2 S\bar{K}J - J\bar{K}\alpha^2 SJ\bar{K} + J\bar{K}J\alpha^2 S_o J\bar{K}J] \cdot e^{\alpha F't} dt\|$$

$$\leq [\|Q\| + (3\|S\| + \|S_o\|)\alpha^2 R^2] \int_o^{+\infty} \|e^{\alpha Ft}\|^2 dt \leq R$$

or by using (D-1) (Appendix D)

$$a\alpha^2 R^2 - \frac{1}{\rho^2 \bar{\pi}} R + q \leq 0 \tag{2.36}$$

which holds because of (2.33), (2.34). The rest is easy. □

It is remarked that if A is diagonalizable, then (2.34) gives $\bar{\lambda} \leq -\epsilon\rho^2$) introduction of $\alpha > 0$ induces no improvement of the result, which is in agreement with the fact that scaling cannot facilitate the existence of solutions of (2.9). In case A has Re $[\lambda(A)] > 0$, we can have results similar

to those of Theorem 2.4 by employing $\alpha < 0$.

The geometric interpretation of Theorem 2.4 is given in Figure 3 and shows simply that if the eigenvalues of A lie on the closed half plane on the left of the line ϵ_2, $(\sigma = -\frac{\epsilon(m,\alpha)}{\alpha})$, for some $\alpha > 0$ then there exists K which solves (2.9). If A is diagonalizable then $-\frac{\epsilon(m,\alpha)}{\alpha} = -\epsilon\rho^2$ and since $\rho \geq 1$, the line ϵ_2 is on the left of ϵ_1. In this case a combination of Theorems 2.2(iv) and 2.4 gives an easily verified sufficient condition for solvability of (2.9).

Finally Theorem 2.3(ii) can be interpreted along the same lines as Theorem 2.2, using Figures 1 and 2, where ϵ' is used instead of ϵ. So, if $\epsilon' \neq 0$, A is diagonalizable and all $\lambda(A)$'s lie in a disk as in Figure 2 with ϵ' in place of ϵ, then (2.9) has a solution and the closed loop matrix $\tilde{A}$ is A.S. If $\epsilon' = 0$ and A is diagonalizable (then $\|Q\| = 0$, $\epsilon = 0$), then if the eigenvalues of A lie in the <u>interior</u> of the disk in Figure 2 the same conclusion holds. The version of Theorem 2.3 with $\alpha < 0$ and $\text{Re}\lambda(A) > 0$ results to $\tilde{A}$ unstable, i.e., $\text{Re}[\lambda(\tilde{A})] > 0$ and is thus of no interest to us.

We close this section with five remarks.

(i) The only assumption on Q_i, R_{ij}'s used in developing the proofs in this section was that R_{11}^{-1}, R_{22}^{-1} exist. Neither $Q_i \geq 0$ or ≤ 0 or $R_{ij} \geq 0$, nor any controllability, observability, or optimality conditions were used.

(ii) If $\bar{Q} = Q + KSK + JKJS_0JKJ \geq 0$ (it suffices R_{12} and $R_{21} \geq 0$) and $\tilde{A}$ is A.S., since

$$K\begin{bmatrix} \tilde{A} & 0 \\ 0 & \tilde{A} \end{bmatrix} + \begin{bmatrix} \tilde{A} & 0 \\ 0 & \tilde{A} \end{bmatrix}' K = -\bar{Q}$$

a standard result in Lyapunov theory yields $K \geq 0$. Since $J_i^* = x_0'K_i x_0$ (see [4]) we will have $J_i^* \geq 0$ for every x_0, as it should be expected in case Q_i, $R_{ij} \geq 0$, $i,j = 1,2$.

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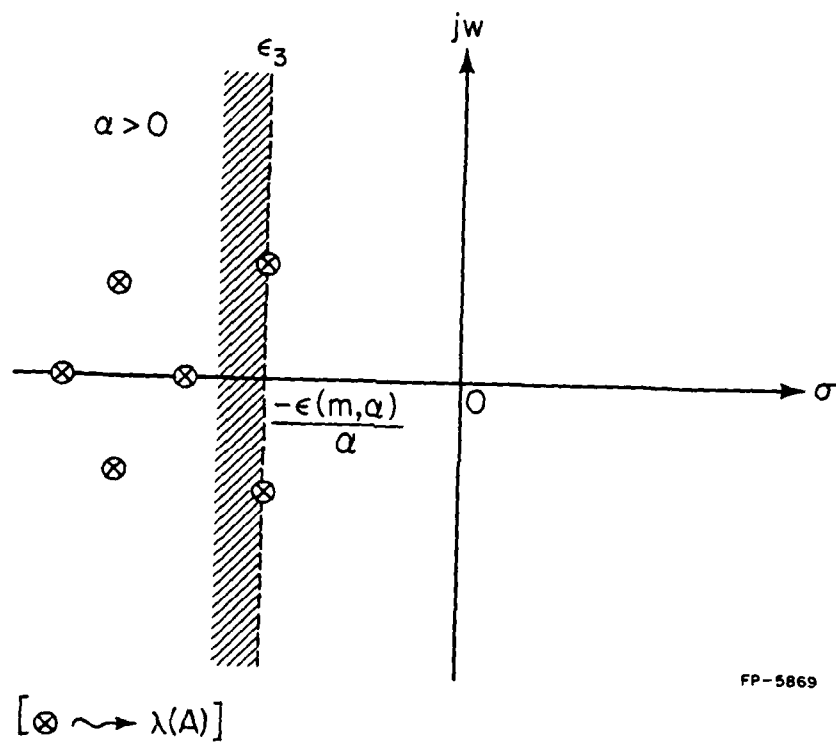
Figure 3. Regions of eigenvalues of A in accordance to Theorem 3.4.

(iii) Consider the single Riccati

$$KA + A'K + Q - KBR^{-1}B'K = 0 \qquad (2.37)$$

where $R > 0$ and A is A.S., with Q not necessarily positive definite. Then Brouwer's Theorem provides results which can be used to easily verify whether the frequency condition of Lemma 5 in [11] holds. It is easy to prove (as in Theorems 2.1, 2.2) that

(i) if there is $\alpha > 0$ such that

$$1 \geq \|I + 2\alpha A\| + 2\alpha \sqrt{\|Q\|\|BR^{-1}B'\|} \qquad (2.38)$$

then (2.37) has a solution K. If in addition

$$\alpha^2 \|S\| R_2 < \|Q\| \qquad (2.39)$$

$$1 > \|I + 2\alpha A\|$$

where

$$R_2 = \frac{1 - \|I + 2\alpha A\| + \sqrt{(1 - \|I + 2\alpha A\|)^2 - 4\|Q\|\|BR^{-1}B'\|}}{2\|S\|} \qquad (2.40)$$

then $A - BR^{-1}B'K$ is A.S.

(ii) If there is an $\alpha > 0$ such that

$$1 \geq \|I + 2\alpha A\| + 2\sqrt{2} \cdot \alpha \sqrt{\|Q\|\|BR^{-1}B'\|} \qquad (2.41)$$

then (2.37) has a solution K and $A - BR^{-1}B'K$ is A.S.

Let now $Q = -C'C \leq 0$, and assume (A,C) observable, (A,B) controllable. Using Lemma 5 of [11], we see that (2.38) and (2.39), or (2.41) imply that

$$I - B'(-jw - A')^{-1}C'C(jw - A)^{-1}B \overset{\geq}{=} 0 \text{ for all real w.} \qquad (2.42)$$

Also, since $K = \int_0^{+\infty} e^{Ft}(Q - KSK)e^{F't}$ and $Q \leq 0$, it will be $K \leq 0$.

(iv) In order to guarantee the fixed point property of $\Phi(K)$, one could have

employed a contraction mapping machinery. Then Φ should map a closed set $D_o \subset X$ into itself and in addition Φ should be Lipschitzian with Lipschitz constant $L, 0 \leq L < 1$ on D_o. But since Φ is quadratic in K D_o should be bounded in order to guarantee that Φ is Lipschitzian there. However this amounts to D_o compact and we could consider D_o a ball B_R w.l.o.g.. So, in order to use the contraction mapping Theorem we should have made assumptions to guarantee $L < 1$ in addition to those made to allow the use of Brouwer's Theorem and this would result in a weaker conclusion.

(iv) The assumptions of Theorem 2.1 guarantee the existence of K_1, K_2 solving (2.4), which lie in B_{R_1}. Thus if the solution of (2.4) is unique it will be in B_{R_1}. If not, then there may be additional solutions K_1, K_2 in B_R, $R > R_1$ which are not in B_{R_1} which solve (2.4).

2.2.3. Extensions

Let us now try to relax the assumption on A to be A.S. Two approaches will be considered. In both of them, we use the solution of an appropriately defined auxiliary problem in order to show existence of solutions to our main problem via Brouwer's theorem.

Consider first the optimal control problem

$$\dot{x} = Ax + [B_1 : B_2] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad x(0) = x_o, \quad t \in [0, +\infty)$$

$$\min \int_0^\infty (x'\tilde{Q}x + [u_1' : u_2']\tilde{R} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}) dt$$

(2.43)

where

$$\tilde{Q} = \frac{Q_1 + Q_2}{2}, \quad \tilde{R} = \begin{pmatrix} \dfrac{R_{11} + R_{21}}{2} & 0 \\ 0 & \dfrac{R_{22} + R_{12}}{2} \end{pmatrix} = \begin{pmatrix} \tilde{R}_1 & 0 \\ 0 & \tilde{R}_2 \end{pmatrix} \tag{2.44}$$

with $\tilde{R}_1$, $\tilde{R}_2 > 0$.[3] Under certain assumptions (controllability-observability and $\tilde{Q} \geq 0$, or see Theorem 2, page 167 in [14] or Remark (iii) in (2.3)) there exists $\tilde{K}$ satisfying

$$0 = \tilde{K}A + A'\tilde{K} + \tilde{Q} - \tilde{K}(\tilde{S}_1 + \tilde{S}_2)\tilde{K} \tag{2.45}$$

with

$$\tilde{S}_1 = B_1 \tilde{R}_1^{-1} B_1', \quad \tilde{S}_2 = B_2 \tilde{R}_2^{-1} B_2'$$

such that

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = - \begin{pmatrix} \tilde{R}_1^{-1} B_1' \bar{K} x \\ \tilde{R}_2^{-1} B_2' \bar{K} x \end{pmatrix} \tag{2.46}$$

solves (2.43), and such that

$$\tilde{\tilde{A}} = A - (\tilde{S}_1 + \tilde{S}_2)\tilde{K} \tag{2.47}$$

i.e., the closed loop matrix for (2.43), is A.S. Let

$$K_1 = \tilde{K} + \Delta_1, \quad K_2 = \tilde{K} + \Delta_2 \tag{2.48}$$

be substituted in (2.4) and by using (2.45) we obtain

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[3] For this to hold it suffices $R_{ii} > 0$, $R_{ij} \geq 0$, $i \neq j$, $i,j = 1,2$.

$$0 = \Delta_1 \widetilde{\widetilde{A}} + \widetilde{\widetilde{A}}' \Delta_1 + \Delta_1 (\widetilde{S}_1 + \widetilde{S}_2 - S_1 - S_2) \widetilde{K} + \widetilde{K}(\widetilde{S}_1 + \widetilde{S}_2 - S_1 - S_2) \Delta_1$$

$$+ \Delta_2 (S_{o1} - S_2) \widetilde{K} + \widetilde{K}(S_{o1} - S_2) \Delta_2 - \Delta_1 S_1 \Delta_1 - \Delta_1 S_2 \Delta_2 - \Delta_2 S_2 \Delta_1$$

$$+ \Delta_2 S_{o1} \Delta_2 + \frac{Q_1 - Q_2}{2} + \widetilde{K}(\widetilde{S}_1 + \widetilde{S}_2 + S_{o1} - S_1 - S_2 - S_2) \widetilde{K}$$

$$\text{(2.49)}$$

$$0 = \Delta_2 \widetilde{\widetilde{A}} + \widetilde{\widetilde{A}}' \Delta_2 + \Delta_2 (\widetilde{S}_1 + \widetilde{S}_2 - S_1 - S_2) \widetilde{K} + \widetilde{K}(\widetilde{S}_1 + \widetilde{S}_2 - S_1 - S_2) \Delta_2$$

$$+ \Delta_1 (S_{o2} - S_1) \widetilde{K} + \widetilde{K}(S_{o2} - S_1) \Delta_1 - \Delta_2 S_2 \Delta_2 - \Delta_2 S_1 \Delta_1 - \Delta_1 S_1 \Delta_2$$

$$+ \Delta_1 S_{o2} \Delta_1 + \frac{Q_2 - Q_1}{2} + \widetilde{K}(\widetilde{S}_1 + \widetilde{S}_2 + S_{o2} - S_1 - S_2 - S_1) \widetilde{K}.$$

where S_i, S_{oi} are given in (2.8).

Let a be as in (2.15) and

$$\Delta = \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix}$$

$$\mu = \|\widetilde{K}(\widetilde{S}_1 + \widetilde{S}_2 - S_1 - S_2)\| + \mu_1$$

$$\mu_1 = \max\{\|\widetilde{K}(S_{o1} - S_2)\|, \|\widetilde{K}(S_{o2} - S_1)\|\} \qquad \text{(2.50)}$$

$$\widetilde{q} = \max\{\|\frac{Q_1 - Q_2}{2} + \widetilde{K}(\widetilde{S}_1 + \widetilde{S}_2 + S_{o1} - S_1 - S_2 - S_2) \widetilde{K}\|,$$

$$\|\frac{Q_2 - Q_1}{2} + \widetilde{K}(\widetilde{S}_1 + \widetilde{S}_2 + S_{o2} - S_1 - S_2 - S_1) \widetilde{K}\|\}$$

$$\widetilde{b} = \|I + 2\alpha \widetilde{\widetilde{A}}\| + 2\alpha \mu$$

$$\widetilde{R}_2 = \frac{1 - \widetilde{b} + \sqrt{(1 - \widetilde{b})^2 - 4\alpha^2 a^2 \widetilde{q}}}{2\alpha^2 a}.$$

The proof of the following Theorem is similar to the proofs of Theorems 2.1 and 2.3.

__Theorem 2.5.__ Let $\overset{\approx}{A}$, μ, $\tilde{q}$ be as in (2.47), (2.50), $\alpha \neq 0$, then

(i) If for some $\alpha > 0$

$$1 \geq \|I + 2\alpha\overset{\approx}{A}\| + 2\alpha(\mu + \sqrt{a\tilde{q}}) \tag{2.51}$$

then there are Δ_1, Δ_2 such that $K_1 = \tilde{K} + \Delta_1$, $K_2 = \tilde{K} + \Delta_2$ solve (2.4) and $\|\Delta_i\| \leq \alpha\tilde{R}_2$.

(ii) If in addition to (2.51)

$$\left.\begin{array}{l} 1 > \tilde{b} \\[2mm] \alpha^2\|S\|\tilde{R}_2^2 < \alpha^2\|S_o\|\tilde{R}_2^2 + 2\alpha\mu_1\tilde{R}_2 + \tilde{q} \end{array}\right\} \tag{2.52}$$

then the closed loop matrix $\tilde{A}$ (as in (2.51)) is A.S.

(iii) If for some $\alpha > 0$

$$\left.\begin{array}{l} 1 \geq \|I + 2\alpha\overset{\approx}{A}\| + 2\alpha(\mu + \sqrt{(a+\|S\|)\tilde{q}}) \\[2mm] 1 > \tilde{b} \\[2mm] q \text{ or } \|S_o\| \text{ or } \mu_1 \neq 0 \end{array}\right\} \tag{2.53}$$

then both (i) and (ii) above hold.

__Proof.__ The equation (2.49) can be written

$$0 = \Delta\tilde{F} + \tilde{F}'\Delta + \psi(\Delta)$$

$$+ \begin{bmatrix} \dfrac{Q_1 - Q_2}{2} + \tilde{K}(\tilde{S}_1 + \tilde{S}_2 + S_{o1} - S_1 - S_2 - S_2)\tilde{K} & 0 \\[4mm] 0 & \dfrac{Q_2 - Q_1}{2} + \tilde{K}(\tilde{S}_1 + \tilde{S}_2 + S_{o2} - S_1 - S_2 - S_1)\tilde{K} \end{bmatrix}$$

where

$$F = \begin{pmatrix} \overset{\approx}{A} & 0 \\ 0 & \overset{\approx}{A} \end{pmatrix}.$$

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Using similar methods as in the proof of Lemma 2.2, we conclude that it suffices to hold

$$a\alpha^2 R + [\|I + 2\alpha\widetilde{\widetilde{A}}\| - 1 + 2\mu\alpha]R + \tilde{q} \le 0 \tag{2.54}$$

in order for (2.49) to have a solution Δ_1, Δ_2 where $\|\Delta_1\|$, $\|\Delta_2\| \le \alpha R$. The rest follows as in the proofs of Theorems 2.1 and 2.3. $\square$

The usefulness of the presented approach is clear in case the game is used to describe a situation where two independent controllers desire to achieve the same objective using slightly different information (Q_i) or control effort (R_{ij}).

Consider now the two independent control problems

$$\dot{x} = Ax + B_1 u_1, \quad x(0) = x_o, \quad t \in [0, +\infty)$$

$$\text{minimize} \int_o^\infty (x'Q_1 x + u_1'R_{11}u_1)dt \tag{2.55}$$

and

$$\dot{x} = Ax + B_2 u_2, \quad x(0) = x_o, \quad t \in [0, +\infty)$$

$$\text{minimize} \int_o^\infty (x'Q_2 x + u_2'R_{22}u_2)dt. \tag{2.56}$$

Under proper assumptions the two Riccati equations

$$0 = A'\overline{K}_1 + \overline{K}_1 A + Q_1 - \overline{K}_1 S_1 \overline{K}_1$$

$$0 = A'\overline{K}_2 + \overline{K}_2 A + Q_2 - \overline{K}_2 S_2 \overline{K}_2 \tag{2.57}$$

have solutions $\overline{K}_1$, $\overline{K}_2$, the $u_1 = -R_{11}^{-1}B_1'\overline{K}_1 x$, $u_2 = -R_{22}^{-1}B_2'\overline{K}_2 x$ solve (2.55) and (2.56), and

$$\overline{A}_1 = A - S_1\overline{K}_1, \quad \overline{A}_2 = A - S_2\overline{K}_2 \tag{2.58}$$

are A.S. Let

$$\overline{F} = \begin{pmatrix} \overline{A}_1 & 0 \\ 0 & \overline{A}_2 \end{pmatrix} = F - S\overline{K}, \quad \overline{K} = \begin{pmatrix} \overline{K}_1 & 0 \\ 0 & \overline{K}_2 \end{pmatrix}$$

$$K_1 = \overline{K}_1 + \Delta_1, \quad K_2 = \overline{K}_2 + \Delta_2 \tag{2.59}$$

$$\Delta = \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix}.$$

Then (2.5) can be written

$$0 = \overline{F}'\Delta + \Delta\overline{F} - \Delta S\Delta - \Delta JS\Delta J - J\Delta SJ\Delta + J\Delta JS_o J\Delta J - \overline{K}JS\overline{K}J - J\overline{K}SJ\overline{K}$$

$$+ J\overline{K}JS_o J\overline{K}J - \overline{K}JS\Delta J - \Delta JS\overline{K}J - J\overline{K}SJ\Delta - J\Delta SJ\overline{K} + J\Delta JS_o J\overline{K}J + J\overline{K}JS_o J\Delta J. \tag{2.60}$$

Let also

$$\overline{\mu} = \|\overline{K}S\| + \|\overline{K}JS_o\| + \|\overline{K}JS\|$$

$$\overline{q} = \|\overline{K}JS\overline{K}J + J\overline{K}SJ\overline{K} - J\overline{K}JS_o J\overline{K}J\|. \tag{2.61}$$

$$\overline{b} = \|I + 2\alpha\overline{F}\| + 2\alpha\overline{\mu}$$

$$\overline{R}_2 = \frac{1 - \overline{b} + \sqrt{(1-\overline{b})^2 - 4\alpha^2 a\overline{q}}}{2\alpha^2 a}$$

Theorem 2.6. Let $\overline{F}$, $\overline{\mu}$, $\overline{q}$ be as in (2.59), (2.61), $\alpha \neq 0$

(i) If for some $\alpha > 0$ it holds

$$1 \geq \|I + 2\alpha\overline{F}\| + 2\alpha[\overline{\mu} + \sqrt{a\overline{q}}] \tag{2.62}$$

then there exist Δ_1, Δ_2 such that $\overline{K}_1 + \Delta_1$, $\overline{K}_2 + \Delta_2$ solve (2.4) and $\|\Delta_i\| \leq \alpha\overline{R}_2$.

(ii) If in addition to (2.62)

$$1 > \overline{b} \tag{2.63}$$

$$\alpha^2\|S\|\overline{R}_2^2 < \alpha^2\|S_o\|\overline{R}_2^2 + 2\alpha(\|\overline{K}JS_o\| + \|\overline{K}JS\|)\overline{R}_2 + \overline{q}$$

then the closed loop matrix $\tilde{A}$ (as in (2.5)) is A.S.

(iii) If for some $\alpha > 0$

$$1 \geq \|I + 2\alpha\bar{F}\| + 2\alpha[\bar{\mu} + \sqrt{a(\bar{q} + \|S\|)}] \tag{2.64}$$

$$1 > b$$

$$\bar{q} \text{ or } \|S_o\| \text{ or } \|\bar{K}JS_o\| + \|\bar{K}JS\| \neq 0$$

then both the conclusions of (i) and (ii) hold.

Proof. Working as in Theorem 2.5, it suffices to hold

$$a\alpha^2 R^2 + [\|I + 2\alpha\bar{F}\| - 1 + 2\alpha\bar{\mu}]R + \bar{q} \leq 0 \tag{2.65}$$

in order for (2.60) to have a solution Δ, where $\|\Delta\| \leq \alpha R$, and so on. □

The usefulness of this approach lies in the fact that the results which pertain to the case where the system is controlled separately by the decision makers can be used to check the existence of the solution when the two decision makers control it jointly and use Nash strategies.

The Theorems 2,2,2.3,2.4 and the interpretations in Figures 1, 2, 3 hold also for the two approaches presented, with the appropriate modifications. For example for Theorems 2.5,2.6, Figures 1, 2 and the first approach one should use $\tilde{\tilde{A}}$, $\mu + \sqrt{a\tilde{q}}$, $\mu + \sqrt{(a + \|S\|)\tilde{q}}$ instead of A, ϵ, ϵ' respectively.

Finally, note that the existence results in all cases developed previously are dependent on the parameter ϵ (or ϵ'). Since ϵ is a function of the weighting matrices and since rescaling of the criteria will affect the weighting matrices it is of interest to point out how this scaling affects the existence results. Nothing changes in the game if we have $J_i' = r_i J_i$ instead of J_i, $r_i > 0$, $i = 1,2$. So considering $r_i Q_i$, $r_i R_{ij}$ instead of Q_i,

R_{ij} we have

$$\epsilon^2 = \max(r_1\|Q_1\|, r_2\|Q_2\|)[3\max(\frac{\|S_1\|}{r_1}, \frac{\|S_2\|}{r_2}) + \max(\frac{\|S_{o1}\|}{r_1}, \frac{\|S_{o2}\|}{r_2})]$$

or

$$\epsilon^2 = \epsilon^2(r) = \max(\frac{\|Q_1\|}{r}, \|Q_2\|) \cdot [3\max(r\|S_1\|, \|S_2\|) + \max(r\|S_{o1}\|, \|S_{o2}\|)]$$

where $r = \dfrac{r_2}{r_1}$. Carrying out the minimization of $\epsilon^2(r)$ with respect to r we find the minimum ϵ^*

$$\epsilon^* = \sqrt{3\max[\|Q_1\| \cdot \|S_1\|, \|Q_2\| \cdot \|S_2\|] + \max[\|Q_1\| \cdot \|S_{o1}\|, \|Q_2\| \|S_{o2}\|]} . \qquad (2.66)$$

The r^* at which $\epsilon(r)$ becomes minimum is given by the following. Let

$$r_\alpha = \min(\frac{\|S_2\|}{\|S_1\|}, \frac{\|S_{o2}\|}{\|S_{o1}\|}) , \quad r_\beta = \max(\frac{\|S_2\|}{\|S_1\|}, \frac{\|S_{o2}\|}{\|S_{o1}\|}) , \quad \bar{r} = \frac{\|Q_1\|}{\|Q_2\|} .$$

If $\bar{r} \leq r_\alpha \leq r_\beta$ then r^* is any point in $[\bar{r}, r_\alpha]$.

If $r_\alpha \leq \bar{r} \leq r_\beta$ then $r^* = \bar{r}$.

If $r_\alpha \leq r_\beta \leq \bar{r}$ then r^* is any point in $[r_\beta, \bar{r}]$.

For ϵ' as in (2.29) the same analysis holds and the optimum ϵ'^* is given by a relation exactly the same as (2.66) but with 4 multiplying the first term instead of 3. We can consider in all of our conditions that ϵ, (ϵ') is given by (2.66). Notice also that a similar procedure will give the minimum values of $\mu + \sqrt{aq}$, see (2.50), (2.61). It is interesting to notice that if $\bar{r} \leq r_\alpha$ then all r: $\bar{r} \leq r \leq r_\alpha$ give the same ϵ^*. Actually, as (2.3) indicates, the existence of solutions for the game should not depend on multiplying J_1 or J_2 by a positive constant. Our conditions have at least preserved this property for an interval $[\bar{r}, r_\alpha]$ or $[r_\beta, \bar{r}]$.

2.3. Finite Horizon: Existence of Closed Loop Nash Strategies and Solutions to Coupled Differential Riccati Equations

2.3.1. Problem Statement

Let us consider the dynamic system

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2, \quad x(0) = x_o, \quad t \in [0, T] \tag{2.67}$$

the two cost functionals

$$J_i(u_1, u_2) = \frac{1}{2} \left\{ x(T)'K_i x(T) + \int_0^T (x'Q_i x + u_i'R_{ii}u_i + u_j'R_{ij}u_j)dt \right\} \tag{2.68}$$

$$i, j = 1,2, \quad i \neq j$$

and the associated Nash game, see [1]. The state x and the strategies u_1, u_2 take values in R^n, R^{m_1} and R^{m_2} respectively. The matrices A, B_i, Q_i, R_{ij}, are real valued piecewise continuous functions of time and of appropriate dimensions. We also assume $K_i = K_i' \geq 0$ constant real matrices, $Q_i(t) = Q_i(t)' \geq 0$, $R_{ij}(t) = R_{ij}(t)' \geq 0$, $R_{ii}(t) > 0$, $\forall t \in [0,T]$, where the time interval [0,T] is assumed fixed.

We restrict the admissible strategies to those which are linear in x, i.e;

$$u_i(t) = L_i(t)x(t), \quad i = 1,2.$$

It can be shown ([1]) that if such an equilibrium Nash pair of strategies exist, it will be given by

$$u_i = - R_{ii}^{-1} B_i'P_i x, \quad i = 1,2, \tag{2.69}$$

where P_1, P_2 satisfy a system of two coupled differential Riccati equations. This system can be written as:

$$-\dot{P} = F'P + PF + Q - PSP - PJSPJ - JPSJP + JPJS_o JPJ \qquad (2.70)$$

$$P(T) = K_o, \qquad t \in [0, T]$$

where

$$F = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$$

$$S = \begin{bmatrix} B_1 R_{11}^{-1} B_1' & 0 \\ 0 & B_2 R_{22}^{-1} B_2' \end{bmatrix}$$

$$S_o = \begin{bmatrix} B_2 R_{22}^{-1} R_{12} R_{22}^{-1} B_2' & 0 \\ 0 & B_1 R_{11}^{-1} R_{21} R_{11}^{-1} B_1' \end{bmatrix} \qquad (2.71)$$

$$Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}$$

$$J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \quad I = n \times n \text{ unit matrix}$$

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$$

The purpose of the present paper is to give sufficient conditions under which (2.70) has a solution over [0, T].

2.3.2. Derivation of the Sufficiency Conditions

By setting

$$\overline{P}(t) = P(T-t) \tag{2.72}$$

we can consider equivalently to (2.70)

$$\dot{\overline{P}} = \overline{F}'\overline{P} + \overline{P}\,\overline{F} + \overline{Q} - \overline{P}\,\overline{S}\,\overline{P} - \overline{P}\,J\,\overline{S}\,\overline{P}\,J - J\,\overline{P}\,S\,J\,\overline{P} + J\,\overline{P}\,J\,\overline{S}_o J\,\overline{P}\,S \tag{2.73}$$

$$\overline{P}(0) = K_o, \qquad t \in [0,\ T]$$

where

$$\overline{X}(t) = X(T-t), \qquad t \in [0,T].$$

If $\overline{P}(t)$ is a solution of (2.73) on $[0, t')$ where $t' \leqslant \hat{T}$, then

$$\|\dot{\overline{P}}(t)\| \leq \beta\|\overline{P}(t)\|^2 + \alpha\|\overline{P}(t)\| + q \tag{2.74}$$

where $\|\cdot\|$ denotes the usual sup norm of a square matrix calculated for fixed t, and

$$\alpha = \max\{2\|A(t)\|\ ;\ 0 \leq t \leq T\}$$

$$\beta = \max\{3\|S(t)\| + \|S_o(t)\|;\ 0 \leq t \leq T\} \tag{2.75}$$

$$q = \max\{\ \|Q(t)\|\ ;\ 0 \leq t \leq T\}$$

The α, β, q are finite due to the piecewise continuity of the matrices and the finiteness of [0, T]. Clearly α, β, q ≥ 0. We assume $\beta \neq 0$, since if $\beta = 0$ then (2.70) is a linear differential equation and the solution exists for T arbitrarily large.

Consider the scalar differential equation

$$\dot{y} = \beta y^2 + \alpha y + q \ , \quad y(0) = y_o, \quad t \geq 0. \tag{2.76}$$

Using Corollary 6.3 page 32 of $[16]^4$, we obtain that:

if $y(0) \geq \|\overline{P}(0)\|$ and $y(t)$ is a solution of (10) on $[0,T]$ then the

solution of (7) exists on $[0,T]$ and $\|P(t)\| \leq y(t)$, $t \in [0,T]$.

We thus conclude that a sufficient condition for the existence of a continuous

solution $P(t)$ of (2.70) over $[0,T]$, is that

$$y(0) \geq \|K_o\| \text{ and } T < t_f \tag{2.77}$$

where $[0, t_f)$ is the maximal interval of existence of the continuous solution

of (2.76).

A straightforward investigation of the behavior of the solution of

(2.76) yields the conditions under which (2.77) is satisfied. We state the results

of this investigation in the form of a Theorem.

Theorem 2.7: Let $\beta \neq 0$ and set

$$\Delta = \alpha^2 - 4\beta q, \qquad \rho_1 = \frac{-\alpha + \sqrt{\Delta}}{2\beta} , \qquad \rho_2 = \frac{-\alpha - \sqrt{\Delta}}{2\beta} .$$

(i) If $\Delta = 0$, and

$$T < \frac{2}{\alpha + 2\beta \|K_o\|} \tag{2.78.1}$$

then the solution of (2.70) exists and

---

^{4} Although Corollary 6.3, page 32 of [16], is stated for the vector case, its
extension to the matrix case is trivial.

$$\|P(t)\| \le \rho_1 + \frac{1}{C - \beta(T-t)}, \qquad C = \frac{2\beta}{\alpha + 2\beta\|K_o\|}, \qquad t \in [0,T]. \tag{2.78.2}$$

(ii) If $\Delta > 0$, $\rho_2 < \rho_1 \le 0$, and

$$T < \frac{1}{\sqrt{\Delta}} \, \ln \left(\frac{\|K_o\| - \rho_2}{\|K_o\| - \rho_1} \right) \tag{2.79.1}$$

then the solution of (2.70) exists and

$$\|P(t)\| \le \frac{\rho_1 - \rho_2 Ce^{\beta(\rho_1-\rho_2)(T-t)}}{1 - Ce^{\beta(\rho_1-\rho_2)(T-t)}} \tag{2.79.2}$$

$$C = \frac{\|K_o\| - \rho_1}{\|K_o\| - \rho_2}, \qquad t \in [0,T].$$

(iii) If $\Delta < 0$, $\rho_1 = k + i\lambda$, $\rho_2 = k - i\lambda$, $k = -\frac{\alpha}{2\beta}$, $\lambda = \frac{\sqrt{|\Delta|}}{2\beta}$, $i = \sqrt{-1}$ and

$$T < \frac{1}{\sqrt{|\Delta|}} \, [\pi - 2 \, \text{Tan}^{-1} \, (\frac{\|K_o\| - k}{\lambda})] \tag{2.80.1}$$

then the solution of (2.70) exists and

$$\|P(t)\| \le k + \lambda \tan (\lambda\beta(T-t) + C) \tag{2.80.2}$$

$$C = \text{Tan}^{-1}(\frac{\|K_o\| - k}{\lambda}), \qquad t \in [0,T]$$

where Tan^{-1} is the inverse tan on $(-\frac{\pi}{2}, \frac{\pi}{2})$.

If $A \equiv 0$, $Q_i \equiv 0$, $K_i = 0$ and B_i, R_{ij} are constant, then case i holds and T can be taken arbitrarily large. $\quad \Box$

2.4. Finite Horizon: Uniqueness of Analytic Nash Strategies for Analytic Differential Games

2.4.1. An Analytic Nash Differential Game

Consider the sets U_1 and U_2 defined as follows

$$I = (t_o', t_f') \subseteq R, \text{ fixed}$$

(2.81)

$$\Sigma = \{S \mid S \subseteq R^n \times I, \text{ S open, connected and projection of S on } I = I\}$$

$$U_i = \{u_i \mid u_i : S_i \to R^{m_i}, \text{ for some } S_i \in \Sigma, u_i \text{ analytic on } S_i\}, i = 1,2. \quad (2.82)$$

U_1 and U_2 will be called the strategy spaces. Consider also the fixed time interval $[\bar{t}_o, t_f] \subseteq I$ and the functions $f : R^n \times R^{m_1} \times R^{m_2} \times I \to R^n$, $g_i : R^n \to R$, $L_i : R^n \times R^{m_1} \times R^{m_2} \times I \to R$, $i = 1,2$, which are analytic everywhere in all their arguments.

For a given $(u_1, u_2) \in U_1 \times U_2$ with $S_1 \cap S_2 \in \Sigma$ we consider the dynamic system

$$\dot{x}(t) = f(x(t), u_1(x(t),t), u_2(x(t),t), t)$$

(2.83)

$$x(t_o) = x_o, \quad (x_o, t_o) \in S_1 \cap S_2, \quad \bar{t}_o \leq t_o \leq t_f.$$

Definition 2.1: A pair $(u_1, u_2) \in U_1 \times U_2$ is called playable at (x_o, t_o) if $S_1 \cap S_2 \in \Sigma$ and the solution of (2.83) exists over $[t_o, t_f]$. (It will necessarily hold: $(x(t), t) \in S_1 \cap S_2$ $t \in [t_o, t_f]$.)

For a related definition of playability see [23]. If (u_1, u_2) is playable at (x_o, t_o) we consider the functionals

$$J_i(u_1, u_2) = g_i(x(t_f)) + \int_{t_o}^{t_f} L_i(x(t), u_1(x(t),t), u_2(x(t),t), t))dt$$

(2.84)

$$i = 1,2.$$

Definition 2:2: A pair $(u_1^*, u_2^*) \in U_1 \times U_2$ is said to be a Nash equilibrium pair

for the Nash game associated with (2.83) and (2.84) on a set $S_o = S \cap (R^n \times [\bar{t}_o, t_f])$ for some $S \in \Sigma$ if and only if:

(i)   it is playable at all $(x_o, t_o) \in S_o$

and

(ii) $J_1(u_1^*, u_2^*) \le J_1(u_1, u_2^*)$, $\forall (u_1, u_2^*) \in U_1 \times U_2$ playable at (x_o, t_o) (2.85)

$J_2(u_1^*, u_2^*) \le J_2(u_1^*, u_2)$, $\forall (u_1^*, u_2) \in U_1 \times U_2$ playable at (x_o, t_o) (2.86)

for all $(x_o, t_o) \in S_o$.

The following theorem concerns the existence and uniqueness of Nash equilibria.

<u>Theorem 2.8</u>: Assume that there exist two analytic[5] functions $\bar{u}_1$ and $\bar{u}_2$, $\bar{u}_i : R^n \times I \times R^n \times R^n \to R^{m_i}$ $i = 1,2$, such that $\bar{u}_1(y,s,q_1,q_2)$ and $\bar{u}_2(y,s,q_1,q_2)$ are the unique global solutions[5] of the minimization problems

$$\min_{u_i \in R^{m_i}} q_i' f(y,u_1,u_2,s) + L_i(y,u_1,u_2,s)$$ (2.87)

where

$$(y,s,q_1,q_2) \in R^n \times I \times R^n \times R^n.$$

Then a necessary and sufficient condition for $(u_1^*, u_2^*) \in U_1 \times U_2$ to be a Nash equilibrium pair is that

$$u_i^*(y,s) = \bar{u}_i(y,s, \frac{\partial V_1(y,s)}{\partial y}, \frac{\partial V_2(y,s)}{\partial y})   \quad i = 1,2$$ (2.88)

---

[5]In relation to this see for example [21], p. 152.

where $V_1(y,s)$, $V_2(y,s)$ are the unique real valued analytic solutions of the system of partial differential equations

$$\frac{\partial V_i}{\partial s} + \frac{\partial V_i'}{\partial y} f(y, \bar{u}_1(y,s, \frac{\partial V_1}{\partial y}, \frac{\partial V_2}{\partial y}), \bar{u}_2(y,s, \frac{\partial V_1}{\partial y}, \frac{\partial V_2}{\partial y}), s)$$

$$+ L_i(y, \bar{u}_1(y,s, \frac{\partial V_1}{\partial y}, \frac{\partial V_2}{\partial y}), \bar{u}_2(y,s, \frac{\partial V_1}{\partial y}, \frac{\partial V_2}{\partial y}), s) = 0 \qquad i = 1,2 \qquad (2.89)$$

with initial values

$$V_i(y, t_f) = g_i(y), \qquad i = 1,2, \qquad \forall y \in R^n. \qquad (2.90)$$

If such (u_1^*, u_2^*) exists, then it is unique in $U_1 \times U_2$.

Proof: Let $(u_1^*, u_2^*) \in U_1 \times U_2$ be a Nash pair. Then the functions

$$\bar{V}_i(y,s) = g_i(x(t_f)) + \int_s^{t_f} L_i(x(t), u_1^*(x(t),t), u_2^*(x(t),t), t)dt \qquad (2.91)$$
$$i = 1,2$$

where

$$\dot{x}(t) = f(x(t), u_1^*(x(t),t), u_2^*(x(t),t), t), x(s) = y, \quad t \in [s, t_f] \qquad (2.92)$$

are analytic in y,s (see [20], p. 44 and [24], p. 87, Theorem 4.3) and are the solutions of (2.89), (2.90). This is true since (2.89), (2.90) are just the Hamilton-Jacobi partial differential equations for the two control problems (2.85) and (2.86) (see [24], p. 83, Theorem 4.1 or [22], Theorem 1). The sufficiency part follows from Theorem 4.4, p. 87 of [24]. The uniqueness of the solution of (2.89)-(2.90) within the analytic class is an immediate consequence of the Cauchy-Kowalewsky theorem (see [25], p. 40). □

One can obtain the system (2.89)-(2.90) of partial differential equations, under assumptions much weaker than ours. Nontheless, the results available concerning existence and uniqueness of solutions of general systems of partial differential equations are complicated. Also, they usually assume boundness of the range spaces of the sought solutions and of the domains of the independent variables, assumptions which we did not make.

2.4.2. The Linear Quadratic Case

Next, we apply Theorem 2.8 to a linear quadratic game. Consider (see [22])

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2 + f(t), \quad x(t_o) = x_o, \quad t \in [t_o, t_f]$$

$$y_1 = C_1 x, \quad y_2 = C_2 x$$

$$J_i = \frac{1}{2} \{ \int_{t_o}^{t_f} [(z_i - y_i)' Q_i (z_i - y_i) + u_i' R_{ii} u_i + u_j' R_{ij} u_j] dt + \qquad (2.93)$$

$$+ x(t_f)' K_{if} x(t_f)' \} \quad , \quad i = 1, 2, \ i \neq j.$$

where A, B_i, f, C_i, Q_i, R_{ij}, z_i are analytic functions of t over all (t_o', t_f') and $Q_i = Q_i' \geq 0$, $R_{ij} = R_{ij}' \geq 0$, $R_{ii} > 0$, $\forall t \in [t_o, t_f]$. $K_{if} = K_{if}' \geq 0$ are constant matrices. All the matrices are assumed to be of appropriate dimensions.

<u>Corollary 2.1</u>. Assume that $K_1(t)$, $K_2(t)$, $g_1(t)$, $g_2(t)$, $\Phi_1(t)$, $\Phi_2(t)$ are solutions of the differential equations

$$\dot{K}_i = K_1 S_1 K_i + K_i S_1 K_1 + K_2 S_2 K_i$$

$$- K_i S_2 K_2 - K_i S_{ij} K_j - K_j S_{ij} K_i$$

$$- A' K_i - K_i A - K_i S_i K_i - C_i' Q_i C_i$$

$$\dot{g}_i = K_i S_1 g_1 + K_1 S_1 g_i + K_i S_2 g_2 + K_2 S_2 g_i \qquad (2.94)$$

$$- \frac{1}{2}(K_i S_i g_i + K_j S_{ij} g_i + K_i S_{ij} g_i) - A' g_i - C_i' Q_i z_i + K_i f$$

$$\dot{\phi}_i = g_1' S_1 g_i + g_2' S_2 g_i - \frac{1}{2} g_i' S_i g_i - g_i S_{ij} g_j + \frac{1}{2} z_i' Q_i z_i - f' g_i$$

$$K_i(t_f) = K_{if}, \ g_i(t_f) = 0, \ \phi_i(t_f) = 0, \ i = 1, 2$$

where

$$S_i = B_i R_{ii}^{-1} B_i' \quad i = 1, 2$$

$$S_{ij} = B_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} B_j', \ i \neq j, \quad i,j = 1, 2$$

$$S_{ij} = 0 \quad i = j$$

Then

$$u_i(x,t) = -R_{ii}^{-1} B_i' [K_i x - g_i], \ i = 1, 2$$

constitute the unique Nash equilibrium pair in $U_1 \times U_2$ for the Nash game associated with (2.93), for any $(x_o, t_o) \in R^n \times [\bar{t}_o, t_f]$.

<u>Proof</u>. K_i, g_i, ϕ_i, are clearly analytic functions of t. The function $V_i(y,s) = \frac{1}{2} x' K_i x - g_i' x + \phi_i$ are solutions of (2.89) and (2.90) (in the form that (2.89) and (2.90) assume for the problem (2.93)). Thus, the previously stated Theorem applies. $\qquad \qquad \square$

CHAPTER 3

LEADER-FOLLOWER STRATEGIES AND NONSTANDARD CONTROL PROBLEMS

3.1. Introduction

In the present Chapter we analyze a Leader-Follower differential game with fixed time interval and initial condition x_0, where the leader has current state information. Thus, the strategy space of the leader is a space of functions whose values at time t depend on $x(t)$, x_0 and t, i.e; $\bar{u}(t) = u(x(t),t;x_0)$. Because $\frac{\partial u}{\partial x}$ appears in the follower's necessary conditions, the leader is faced with a nonstandard control problem. This problem is solved here. Actually we solve a more general class of such nonstandard control problems, of which the leader's problem is a special case.

The structure of this Chapter is the following. In 3.2 a two-level L-F differential game is introduced. This game leads to the consideration of a nonstandard control problem which is studied in 3.3. In 3.4 we use the results of 3.3 to study further the game of 3.2 and in particular we work out a linear quadratic L-F game. In 3.5 the relation of the L-F game to the Principle of Optimality is investigated.

Two Appendices to this Chapter are given at the end of the Thesis.

3.2. A Leader-Follower Game

In this section we introduce a two-level L-F game and show how it leads us to the consideration of a nonstandard control problem. This nonstandard control problem falls into the general class to be considered in 3.3.

Let

$$U = \{u \mid u: R^n \times [t_o, t_f] \to R^{m_1}, \ u(x,t) \in R^{m_1} \text{ for } x \in R^n \text{ and } t \in [t_o, t_f],$$

$$u_x(x,t) \text{ exists and } u(x,t), \ u_x(x,t) \text{ are continuous in } x \text{ and}$$

$$\text{piecewise continuous in } t\} \tag{3.1}$$

$$V = \{v \mid v: [t_o, t_f] \to R^{m_2}, \ v \text{ is piecewise continuous in } t\}. \tag{3.2}$$

Consider the dynamic system

$$\dot{x}(t) = f(x(t), \bar{u}(t), \bar{v}(t), t), \quad x(t_o) = x_o, \ t \in [t_o, t_f] \tag{3.3}$$

and the functionals

$$J_1(u,v) = g(x(t_f)) + \int_{t_o}^{t_f} L(x(t), \bar{u}(t), \bar{v}(t), t) dt \tag{3.4}$$

$$J_2(u,v) = h(x(t_f)) + \int_{t_o}^{t_f} M(x(t), \bar{u}(t), \bar{v}(t), t) \tag{3.5}$$

where $u \in U$, $v \in V$, x is the state of the system, assumed to be a continuous function of t and piecewise in C^1 w.r. to t, $x : [t_o, t_f] \to R^n$, and the functions $f : R^n \times R^{m_1} \times R^{m_2} \times [t_o, t_f] \to R^n$, $g, h : R^n \to R$, $L, M : R^n \times R^{m_1} \times R^{m_2} \times [t_o, t_f] \to R$, are in C^1 w.r. to the x,u,v arguments and continuous in t. The u and v are called strategies and are chosen from U and V which are called the strategy spaces, by the two players, the leader and the follower respectively. With the given definitions, for each choice of u and v, the behavior of the dynamic system is unambiguously determined, assuming of course, that for the selected pair (u,v) the solution of the differential equation (3.3) exists over $[t_o, t_f]$.

Let us assume that a LF equilibrium pair $(u^*, v^*) \in U \times V$ exists. For fixed $u \in U$, Tu is determined by the minimization problem

minimize $J_2(u,v)$

subject to: $v \in V$ (3.6)

$$\dot{x} = f(x, u(x,t),v,t), \quad x(t_o) = x_o, \quad t \in [t_o,t_f]$$

and thus, applying the Minimum Principle we conclude that for $v \in V$ to be in Tu, there must exist a function $p : [t_o,t_f] \to R^n$ such that

$$\dot{x} = f(x,u,v,t) \tag{3.7.1}$$

$$M_v + f_v p = 0 \tag{3.7.2}$$

$$-\dot{p} = M_x + u_x M_u + (f_x + u_x f_u)p \tag{3.7.3}$$

$$x(t_o) = x_o, \quad p(t_f) = \frac{\partial h(x(t_f))}{\partial x} . \tag{3.7.4}$$

We further assume that U is properly topologized. Conditions (3.7) define a set valued mapping $T' : U \to V$. By using the nature of the defined U and V and the fact that (3.7) are necessary but not sufficient conditions it is easily proven that

(i) $Tu \subseteq T'u$

(ii) $J_2(u,v') \geq J_2(u,v) \quad \forall : v' \in T'u, \ v \in Tu,$

(iii) $T'u^* \cap Tu^* \supseteq \{v^*\} \neq \emptyset.$

Notice that $J_2(u,v)$ takes one value for given u and any $v \in Tu$, while $J_2(u,v'), \ v' \in T'u$ does not necessarily do so. We assume now the following.

Assumption (A) :

$$J_1(u,v') \geq J_1(u,v) \text{ for } v' \in T'u, \quad v \in Tu, \quad u \in U_N^* \tag{3.8}$$

where U_N^* is a n.b.d. of u^* in U.

For (A) to hold it suffices for example: $T = T'$ on U_N^*.[1] We conclude

that if (A) holds, then u^* is a local minimum of the problem

minimize $J_1(u,v)$

subject to: $u \in U$, $v \in T'u$

or equivalently

minimize $J_1(u,v)$

subject to: $u \in U$, $v \in V$ $\qquad\qquad$ (3.9)

$$\dot{x} = f(x,u,v,t) \qquad\qquad (3.10.1)$$

$$-\dot{p} = M_x + u_x M_u + (f_x + u_x f_u)p \qquad\qquad (3.10.2)$$

$$M_v + f_v p = 0 \qquad\qquad (3.10.3)$$

$$x(t_o) = x_o, \quad p(t_f) = \frac{\partial h(x(t_f)}{\partial x} \quad . \qquad\qquad (3.10.4)$$

The problem (3.9) is a nonstandard control problem of the type to be considered

in the next section, since the partial derivative of the control u w.r. to x

appears in the constraints (3.10) which play the role of the system differen-

tial equations and state control constraints, with new state $(x',p')'$.

Notice that the leader uses only $x(t)$ and t in evaluating $u(x(t),t)$ and not

the whole state $(x',p')'$; i.e., the value of u at time t is composed in a

partial feedback form with respect to the state $(x',p')'$; (recall the output

feedback in contrast to the state feedback control laws). If one were

concerned with a LF game composed of N (≥ 2) hierarchical decision levels

([32] , [33]), then the leader would face a nonstandard control problem where

---

[1] See Appendix E.

the N-th partial of u with respect to x would appear.

We will assume that the state-control constraint (3.10.3) can be solved for v over the whole domain of interest to give

$$v = S(x,p,u,t) \qquad (3.11)$$

where S is continuous and in C^1 w.r. to x and p. This assumption holds in many cases, as for example in the linear quadratic case to be considered in section 3.4. In any case, direct handling of the constraint (3.10.3) by appending it, or assumption of its solvability in v, does not seem to be the core of the matter from a game point of view. However the following remark is pertinent here. Assume that we allow $v \in \bar{V}$, where

$$\bar{V} = \{v \mid v : R^n \times [t_o, t_f] \to R^{m_2}, \ v(x,t) \text{ piecewise continuous}$$
$$\text{in t and Lipschitzian in x} \quad \text{where } x \in R^n \text{ and } t \in [t_o, t_f]\} \qquad (3.12)$$

instead of $v \in V$. The assumption of solvability of (3.10.3) will again give

$$v(x,t) = S(x,p,u,t). \qquad (3.13)$$

Since v(x,t) will be substituted in the rest of (3.10) with S(x,p,u,t) from (3.13), the leader will be faced with exactly the same problem as after substituting v(t) with S from (3.11). Therefore, no additional difficulty arises if one allows $\bar{V}$ instead of V and assumes solvability of (3.10.3)

Substituting v from (3.11) to (3.10) we obtain

$$\text{minimize. } J(u) = g(x(t_f)) + \int_{t_o}^{t_f} \tilde{L}(x,p,u,t)dt$$
$$u \in U$$

subject to:

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} F_1(x,p,u,t) \\ F_{21}(x,p,u,t) + u_x F_{22}(x,p,u,t) \end{bmatrix} \qquad (3.14)$$

$$x(t_o) = x_o, \quad p(t_f) = \frac{\partial h(x(t_f))}{\partial x}$$

where $\tilde{L}$, F_1, F_{21}, F_{22} stand for the resulting composite functions. Problem (3.14) is a nonstandard control problem like the one treated in Section 3.3 where $(x',p')'$ is the state of the system.

Besides the procedure described above which leads to the consideration of the problem (3.14), there are other cases in which such problems arise. For example, in a control problem where the state x is available, stochastic disturbances are present, and the time interval $[t_c, t_f]$ is very large, synthesis of the control law as a function of x and t is preferable over a synthesis not using x (open loop). In addition, u_x might be penalized in the cost function or be subjected to bounds of the form $|u_x(x(t),t)| \leq K$, $t \in [t_o, t_f]$, where $K \geq 0$ is a constant.

3.3. A Nonstandard Control Problem

Consider the dynamic system described by

$$\dot{x}(t) = f(x(t), u^1(h^1(x(t),t),t), u^2(h^2(x(t),t),t), \ldots, u^m(h^m(x(t),t),t),$$

$$u_x^1(h^1(x(t),t),t), \ldots, u_x^m(h^m(x(t),t),t),t) \qquad (3.15)$$

$$x(t_o) = x_o, \quad t \in [t_o, t_f]$$

and the functional

$$J(u) = g(x(t_f)) + \int_{t_o}^{t_f} L(x(t), u^1(h^1(x(t),t),t), \ldots, u^m(h^m(x(t),t),t),$$

$$u_x^1(h^1(x(t),t),t), \ldots, u_x^m(h^m(x(t),t),t),t)dt \qquad (3.16)$$

where the functions $f : R^{n+m+mn+1} \to R^n$, $L : R^{n+m+mn+1} \to R^n$, $h^i \ "R^{n+1} \to R^{q_i}$

$i = 1, \ldots, m$, $g : R^n \to R$ are continuous in all arguments and in C^1 with respect to

the x, u, u_x^i. The functions $h^i : R^{n+1} \to R^{q_i}$, are continuous, and in C^2 w.r. to x.[2]

The solution x of (3.15) is assumed to be continuous and piecewise in

C^1 w.r. to t. The time interval $[t_o, t_f]$ is considered fixed w.l.o.g. (See [35],

page 27). We want to find a function u where

$$u = \begin{bmatrix} u^1 \\ \cdot \\ \cdot \\ u^m \end{bmatrix}$$

$$u^i : R^{q_i} \times [t_o, t_f] \to R, \qquad i = 1, \ldots, m$$

$u_x^i(h^i(x,t),t)$ exists and $u^i(h^i(x,t),t)$, $u_x^i(h^i(x,t),t)$ are continuous in x

and piecewise continuous in t, for $x \in R^n$, $t \in [t_o, t_f]$, $i = 1, \ldots, m$ so as to

minimize $J(u)$. We denote by $\bar{U}$ the set of all such u's. Therefore the

problem under investigation is

$$\text{minimize } J(u)$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.17)$$
$$\text{subject to } u \in \bar{U} \text{ and } (3.15)$$

---

[2] The restriction $h \in C^2$ w.r. to x is somewhat strong. For example, the case $h(x,t) = x$ if to $\leq t \leq t_1$, $h(x,t) = 0$ if $t_1 < t \leq t_f$, i.e. the state is available only during a part of the $[t_o, t_f]$ is not included. Nonetheless, it can be approximated arbitrarily close by a C^2 function, like any function which is only piecewise C^2. Consequently, from an engineering point of view, $h \in C^2$ w.r. to x is not a serious restriction.

This problem is posed for a fixed time interval $[t_o, t_f]$ and a fixed initial condition $x(t_o) = x_o$. Therefore the solution u^*, if it exists, will in general be a function of t_o, t_f, x_o, in addition to being a function of $h(x,t)$, t, but we do not show this dependence on t_o, t_f, x_o explicitly.

We use the notation

$$f_u = \begin{bmatrix} \dfrac{\partial f}{\partial u^1} \\ \vdots \\ \dfrac{\partial f}{\partial u^m} \end{bmatrix}, \ m \times n \text{ matrix}, \quad L_u = \begin{bmatrix} \dfrac{\partial L}{\partial u^1} \\ \vdots \\ \dfrac{\partial L}{\partial u^m} \end{bmatrix}, \ m \times 1 \text{ vector}$$

$$f_i = \frac{\partial f}{\partial (u^i_x)}, \quad n \times n \text{ matrix}, \quad i = 1,\ldots,m$$

$$L_i = \frac{\partial L}{\partial (u^i_x)}, \quad n \times n \text{ vector}, \quad i = 1,\ldots,m$$

(3.18)

$$u^i_j = \frac{\partial u^i(y^i,t)}{\partial y^i_j}, \ y^i = (y^i_1,\ldots,y^i_j,\ldots,y^i_{q_i})' \in R^{q_i}$$

$$h^i = (h^i_1,\ldots,h^i_j,\ldots,h^i_{q_i})', \quad i = 1,\ldots,m, \ j = 1,\ldots,q_i$$

$$u^i_{y^i} = (u^i_1,\ldots,u^i_{q_i}), \quad u^i_{y^i y^i} = q_i \times q_i \text{ Hessian}$$

$$u^i_x = \frac{\partial h^i(x,t)}{\partial x} \cdot \left. \frac{\partial u^i(y^i,t)}{\partial y^i} \right|_{y^i = h^i(x,t)}, \quad n \times 1 \text{ vector} \quad i = 1,\ldots,m$$

$$u_x = [u^1_x : \cdots : u^m_x], \quad n \times m \text{ matrix}.$$

It should be pointed out that the arguments used in Classical Control Theory for showing that for the fixed initial point case, it is irrelevant for the optimal trajectory and cost whether the control value at time t is composed by using x (t) and t or only t,[3] do not apply here in general. If $u|_t$ = u (t), t $\in$ [t_o, t_f], then u_x = 0 and this changes the structure of problem (8). Consideration of variations of u_x is also needed and this was where the previous researchers stopped, see [29]. This problem is successfully treated here. We provide two different ways of doing that, the first of which is based on an extension (Lemma 3.1) of the so-called "fundamental lemma" in the Calculus of Variations (see [36]).

The following theorem provides necessary conditions for a function u $\in$ $\overline{U}$ to be a solution to the problem (3.17) in a local sense; (we assume that $\overline{U}$ is properly topologized). It is assumed in this theorem that the optimum u^* has strong differentiability properties, an assumption which will be relaxed later, in Theorem 3.2. The proof of this theorem is based on the following Lemma.

<u>Lemma 3.1:</u>   Let M : [t_o,t_f] $\rightarrow$ R^m, N_i : [t_o,t_f] $\rightarrow$ R^n, i = 1,...,m, y : [t_o,t_f] $\rightarrow$ R^n, be continuous functions, such that

$$\int_{t_o}^{t_f} M'(t)\varphi(y(t),t)dt + \sum_{i=1}^{m} \int_{t_o}^{t_f} N_i'(t)\varphi_y^i(y(t),t)dt = 0$$

for every continuous function φ : R^n x [t_o,t_f] $\rightarrow$ R^n, where φ = (φ^1,...,φ^m)', and φ is in C^1 w.r. to y. Then M, N_1,...,N_m are identically zero on [t_o,t_f].

---

[3]This holds if i) the set of the admissible closed loop control laws contain the set of the admissible open-loop control laws and ii) if u^* is an optimal closed loop control law generating an optimal trajectory x*(t), then v*(t) $\overset{\Delta}{=}$ u*(x*(t),t) is an admissible open loop control law.

<u>Proof:</u> The choice $\varphi_i = (0,\ldots,0,\varphi^i,0,\ldots,0)'$, $\varphi^i: [t_o,t_f] \to R, \varphi^i$ continuous in t, $i = 1,\ldots,m$, yields $M \equiv 0$ on $[t_o,t_f]$. Since $M \equiv 0$, the choice $\bar{\varphi}_i = (0,\ldots,y'\psi,0,\ldots,0)'$, $\varphi^i = y'\psi$, where $\psi = (\psi_1,\ldots,\psi_n)'$, $\psi : [t_o,t_f] \to R^n$, ψ continuous in t, results in $\int_{t_o}^{t_f} N_i'(t)\psi(t)dt = 0$, for every such ψ, and thus $N_i \equiv 0$ on $[t_o,t_f]$ is proven in the same way as $M \equiv 0$ was proven. $\qquad\square$

The conclusion of the above lemma holds even if the restriction $\varphi^i(x,t) = y_1^{k_{1i}} \ldots y_n^{k_{ni}} \cdot t^{\lambda_i}$ is imposed, where $k_{1i},\ldots,k_{ni},\lambda_i$ are nonnegative integers, since the polynomials are dense in the space of measurable functions on $[t_o,t_f]$.

<u>Theorem 3.1 :</u> Let $u^* \in \bar{U}$ be a solution of (3.17) which gives rise to a trajectory $\Gamma_1 = \{(x^*(t),t) | t \in [t_o,t_f]\}$, such that $u_{y^i}^i$ are in C^1 w.r.to x in a n.b.d. of $\{(h^i(x^*(t),t),t), t \in [t_o,t_f]\}$. Then there exists a function $p : [t_o,t_f] \to R^n$ such that

$$-\dot{p}(t) = L_x + f_x p + \sum_{i=1}^{m} \sum_{j=1}^{q_i} u_j^i \nabla_{xx} h_j^i (L_i + f_i p) \qquad (3.19)$$

$$L_u + f_u p = 0 \qquad (3.20)$$

$$\nabla_x h^{i'} (L_i + f_i p) = 0, \quad i = 1,\ldots,m \qquad (3.21)$$

$$p(t_f) = \frac{\partial g(x(t_f))}{\partial x} \qquad (3.22)$$

hold for $t \in [t_o,t_f]$, where all the partial derivatives are evaluated at

$$x^*(t), \quad u^{i*}(h^i(x^*(t),t),t), \quad u_x^{i*}(h^i(x^*(t),t),t),t.$$

The proof of this Theorem by using variational techniques and Lemma 3.1 is simple but lengthy. For the sake of completeness, we give it in Appendix F.

We give now a different derivation of the results of Theorem 3.1, under weaker assumptions, which provides an interpretation for them and at the same time an extension of the region of their validity. Let

$$\overline{U}_k = \{\overline{u} \mid \overline{u} : [t_o, t_f] \to R^k, \ \overline{u} \text{ piecewise continuous}\} \qquad (3.23)$$

Consider the problem

$$\text{minimize } J(\overline{u}, \overline{u}_1, \ldots, \overline{u}_m) = g(x(t_f)) + \int_{t_o}^{t_f} L(x, \overline{u}, \nabla_x h^1(x,t)\overline{u}_1, \ldots, \nabla_x h^m(x,t)\overline{u}_m, t)dt$$

$$\text{subject to } \dot{x} = f(x, \overline{u}, \nabla_x h^1(x,t)\overline{u}_1, \ldots, \nabla_x h^m(x,t)\overline{u}_m, t), \quad x(t_o) = x_o, \quad t \in [t_o, t_f]$$

$$\overline{u} \in \overline{U}_m, \quad \overline{u}_i \in \overline{U}_{q_i}, \quad i = 1, \ldots, m. \qquad (3.24)$$

Clearly, if J_1^*, J_2^* are the infima of (3.17) and (3.24) respectively, it will be $J_1^* \leq J_2^*$. Also, if $\overline{u} = (\overline{u}^1, \ldots, \overline{u}^m)'$, $\overline{u}_1, \ldots, \overline{u}_m$ solve (3.24) and give rise to $\underline{x}(t)$, then a $u = (u^1, \ldots, u^m)' \in \overline{U}$ with

$$\begin{bmatrix} u^1(h^1(\underline{x}(t),t),t) \\ \vdots \\ u^m(h^m(\underline{x}(t),t),t) \end{bmatrix} = \overline{u}(t), \quad u_x^i(h^i(\underline{x}(t),t),t) = \nabla_x h^i(\underline{x}(t),t)\overline{u}_i(t)$$

$$i = 1, \ldots, m \qquad (3.25)$$

results in $J_2(u) = J(\overline{u}, \overline{u}_1, \ldots, \overline{u}_m)$ and gives rise to the same $\underline{x}(t)$. However, such $u \in \overline{U}$ does exist. For example we set

$$u^i(h^i(x,t),t) = a_i'(t) \, h^i(x,t) + b_i(t) \tag{3.26}$$

where

$$a_i(t) = \bar{u}_i(t) \tag{3.27}$$

$$b_i(t) = \bar{u}^i(t) - a_i'(t) h^i(\underline{x}(t),t) \tag{3.28}$$

$$i = 1,\ldots,m$$

This u satisfies (3.25). Thus, problems (3.24) and (3.17) are actually equivalent in the sense that for each given (x_o, t_o) they have the same optimal trajectories and costs and their optimal controls are related by (3.25).

The conditions of Theorem 3.1 are now directly verified to be the necessary conditions for problem (3.24), where one should use $\bar{u}$ and $\bar{u}_i$ in place of u and $u^i_{y^i}$ respectively. More importantly, the conditions of Theorem 3.1 hold if one considers simply $u^* \in \bar{U}$, without assuming that $u^{i*}_{y^i}$ is in C^1 w.r. to x in a n.b.d. of $\{(h^i(x^*(t),t),t), t \in [t_o, t_f]\}$. This weakens the strong differentiability property of u^* assumed in Theorem 3.1 . The relative independence of u, $u^i_{y^i}$, was exploited in proving Theorem 3.1 , when the special form of the perturbation $\varphi(y,t)$, $y'\psi(t)$ (see proof of Lemma 3.1), sufficed to conclude (3.20) and (3.21). This independence of u and $u^i_{y^i}$ was taken a priori into consideration, when problem (3.24) was formulated. Clearly, even if higher order partial derivatives of u w.r. to

x appear in f and L, or if $u, u^i_{y^i}$ are restricted to take values within certain closed sets, the equivalence of the corresponding problems (3.17) and (3.24) holds again (with appropriate modifications of the definitions of $\bar{U}$, $\bar{U}_k$, f and L). We formalize the discussion above in the following theorem.

<u>Theorem 3.2</u>: Let $u^* \in \bar{U}$ be a solution to the problem

$$\text{minimize } J(u) = g(x(t_f)) + \int_{t_o}^{t_f} L(x, u, u^i_x, \ldots, u^m_x, t) dt \qquad (3.29)$$

$$\text{subject to: } \dot{x} = f(x, u, u^i_x, \ldots, u^m_x, t), \quad x(t_o) = x_o, \quad t \in [t_o, t_f]$$

$$u \in V, \ (u^1(h^i(x(t), t), t), \ldots, u^m(h^m(x(t), t), u^1_{y^1}(h^1(x(t), t), t)', \ldots,$$

$$u^m_{y^m}(h^m(x(t), t), t)') \in V_o \qquad (3.30)$$

where $V_o \subseteq R^{m+nm}$ is closed. Then there exists

$$p: [t_o, t_f] \to R^n \text{ such that}$$

$$-\dot{p} = L_x + f_x p + \sum_{i=1}^{m} \sum_{j=1}^{q_i} u^i_j \nabla_{xx} h^i_j (L_i + f_i p) \qquad (3.31)$$

$L(x^*(t), u^{1*}(h^1(x^*(t), t), t), \ldots, u^{m*}(h^m(x^*(t), t), t), u^{1*}_x(h^1(x^*(t), t), t),$

$\ldots, u^{m*}_x(h^m(x^*(t), t), t), t) +$

$+ f'(x^*(t), u^{1*}(h^1(x^*(t), t), t), \ldots, u^{m*}(h^m(x^*(t), t), t), u^{1*}_x(h^1(x^*(t), t), t),$

$\ldots, u^{m*}_x(h^m(x^*(t), t), t) \cdot p(t) \leq$

$$\leq L(x^*(t), q^1_o, \ldots, q^m_o, \nabla_x h^1(x^*(t), t) q_1, \ldots, \nabla_x h^m(x^*(t), t) q_m, t)$$

$$+ f(x^*(t), q^1_o, \ldots, q^m_o, \nabla_x h^1(x^*(t), t) q_1, \ldots, \nabla_x h^m(x^*(t), t) q_m, t)$$

$$\forall (q^1_o, \ldots, q^m_o, q'_1, \ldots, q'_m) \in V_o.$$

(3.32)

$$p(t_f) = \frac{\partial g(x^*(t_f))}{\partial x} \tag{3.33}$$

for $t \in [t_o, t_f]$. □

It is remarkable that the established equivalence of the problems (3.17) and (3.24) refers to the optimal trajectories, costs and control values. It does not refer to any other properties, such as sensitivity, for example. It is thus possible, that different realizations of $u^i(h^i(x,t),t)$ other than (3.26) may enjoy sensitivity or other advantages. The following proposition provides information for tackling such problems.

Proposition 3.1.

(i)   If u and v are elements of U, both satisfying (3.25), so does
     $\lambda u + (1-\lambda)u$, $\forall \lambda \in R$.

(ii) Let m=1, $h^1(x,t) = x_1$ and $\bar{x}_1, \bar{u}, \bar{u}_1$ be scalarvalue functions of
     $t, t \in [t_o, t_f]$. Then the function
     $$u(x,t) = e^{x_1(x_1 - \bar{x}_1(t))} \bar{u}(t) + [\bar{u}_1(t) - \bar{x}_1(t)\bar{u}(t)] \cdot [x_1 - \bar{x}_1(t)]$$

     satisfies $u(\bar{x}(t),t) = \bar{u}(t)$, $u_x(\bar{x}(t),t) = \bar{u}_1(t)$

(iii) Let $\bar{x}, \bar{u}, \bar{u}_1$ be as in (ii). Assume that the scalar valued
     functions $u(x,t)$, $v(x,t)$ satisfy $u(\bar{x}(t),t) = v(\bar{x}(t),t) =$
     $\bar{u}(t)$ and $u_x(\bar{x}(t),t) = v_x(\bar{x}(t),t) = \bar{u}_1(t)$. Then so do the
     functions $\frac{2uv}{u+v}$, $\sqrt{uv}$, $\sqrt{\frac{u^2+v^2}{2}}$, assuming that u and v are properly
     behaved. □

The proof of this proposition is a matter of straightforward verification. The assumption in parts (ii) and (iii) for scalar valued quantities actually induces no loss of conceptual generality, since it can be abandoned at the expense of increased complexity of the corresponding expressions

of course.

The nonuniqueness of the solution u to problem (3.17) is obvious in the light of (3.25) and Proposition 3.2. Nonetheless, this nonuniqueness is a nonuniqueness in the representation of u^i as a function of h^i and t, while $u|_t$, $u^i_{y^i}|_t$ are the same for all these representations. The non-uniqueness of $u|_t$, $u^i_{y^i}|_t$, if any, can be characterized in terms of the possible nonuniqueness of the $a_i(t)$, $b_i(t)$ (see (3.26)), where one, w.l.o.g, restricts u^i to affine in h^i controls.

One very basic difference between problems (3.17) and (3.24) is the following. It is clear that the principle of optimality holds for both of these problems, in the sense that the last piece of each optimal trajectory is optimal. The existence of a closed loop control law $(\bar{u}(x,t), \bar{u}_1(x,t), \ldots, \bar{u}_m(x,t)$ which results in an optimal solution to problem (3.24) for every initial point (x_o, t_o) in a subset of R^{n+1} is guaranteed under certain assumptions, see [24]. A corresponding statement does not hold for problem (3.17), i.e. in general there do not exist functions u^i of $h^i(x,t)$ and t such that $u = (u^1, \ldots, u^m)'$ is an optimal solution to problem (3.17) for every initial point (x_o, t_o) in a subset of R^{n+1}. This can be easily seen to hold by the following argument. Let such u exist. Then,

$$(u^1(h^1(x,t),t), \ldots, u^m(h^m(x,t),t), u^1_{y^1}(h^1(x,t),t)', \ldots, u^m_{y^m}(h^m(x,t),t)')'$$

is a closed loop control law for problem (3.24). This implies that there must exist a solution $(\bar{u}, \bar{u}_1, \ldots, \bar{u}_m)$ with $\bar{u} = (\bar{u}^1, \ldots, \bar{u}^m)$ of the partial differential equation of Dynamic Programming associated with problem (3.24) which satisfies $\bar{u}^i(x,t) = u^i(h^i(x,t),t)$ and $\frac{\partial \bar{u}^i(x,t)}{\partial x} = \nabla_x h^i(x,t) \cdot \bar{u}_i(x,t)$, $i = 1, \ldots, m$, which

is not in general true.[4] This difference between problems (3.17) and (3.24) empha-
sizes the fact that their equivalence holds in a restricted fashion, i.e. for each
initial point considered independently and not in a global fashion, like a
closed loop control law treats the initial points.

Two final remarks before entering the next section are
pertinent here. First, that the established equivalence of the problems
(3.17) and (3.24) reduces all questions of existence, uniqueness, controllability
and of sufficiency conditions for problem (3.17) to the corresponding ones
for (3.24). Any problem of the form (3.17) where terminal constraints and
control constraints are present can be solved and necessary and sufficient
conditions can be written down in as much as this can be done for the problem
(3.24) with the corresponding constraints considered in addition. Second,
Theorem 3.2 still holds if instead of the initial condition
$x(t_o) = x_o$, it is given: $x^\alpha(t_o) = x_o^\alpha$ and $x^\beta(t_f) = x_f^\beta$, where $x = (x^{\alpha'}, x^{\beta'})'$.
In this case, (3.33) is modified to

$$p^\alpha(t_f) = \frac{\partial g(x^\alpha(t_f))}{\partial(x^\alpha)} \quad \text{and} \quad p^\beta(t_o) = \frac{\partial h(x^\beta(t_o))}{\partial(x^\beta)} \tag{3.34}$$

where the more general cost functional

$$J = g(x^\alpha(t_f)) + h(x^\beta(t_o)) + \int_{t_o}^{t_f} L(x,u,t)dt \tag{3.35}$$

is considered (see [35]).

---

[4] Imposing the condition $\frac{\partial \bar{u}i}{\partial x} = \nabla_x h^i \cdot \bar{u}_i$, where $\bar{u}^i(x_1 t), \bar{u}_i(x_1 t)$ are given in
terms of $\frac{\partial V(x,t)}{\partial x}$, where $V(x_1 t)$ is the value function for the control problem
(3.24) results in a condition that V must satisfy in addition to being a solution
of the dynamic programming partial differential equation. This procedure can
be used to single out a class of control problems (3.17) where a closed loop
control law exists.

3.4. Solution of the Leader Follower Game

In this section we analyze the LF Game of Section 3.2 by
using the results of Section 3.3. In particular, we work out a
Linear Quadratic LF Game, where the leader is paralized for u_x^i
as well.

Let us consider the LF Game of Section 3.2. In this case,
the h^i's for the leader (u), are

$$h^i((x,p),t) = [\, I_{nxn} \quad 0_{nxn}\,] \quad \{\begin{smallmatrix}x\\p\end{smallmatrix}\} = x, \; i=1,\ldots,m_1$$

and the h^i's for the follower (v) are identically zero. Different h^i's may
be used to model different information structures in terms of $x(t)$, and t
available to the leader and follower at time t. Thus Theorem 3.2 is
applicable and can be used for writing down the leader's necessary
conditions. From the results of the previous section, we conclude that the
solution for the leaders u -if it exists -is not unique. It is interesting
to notice that (3.26) implies that the leader has nothing to lose if he commits
himself to an affine in x, time varying strategy. With such a commitment,
the leader does not deteriorate his cost, does not alter the optimal
trajectory, and also the follower's optimal cost is not affected. More
noteworthy is that the affine choice for the leader can be made even if
f, L, M are nonlinear and u, u_x^i are constrained to take values in given
closed sets. In addition, v may be constrained to take values in a given
closed set in which case (3.10.3) should be substituted by an appropriate
inequality. In accordance with the discussion in the previous section, we
have that in general there does not exist a strategy u (x,t) which is optimal
for every initial point (x_o,t_o) in a subset of R^{n+1}.

Let $\lambda = (\lambda_1', \lambda_2')'$ denote the adjoint variable for problem (3.14) with λ_1, λ_2 corresponding to x and p respectively. Then, condition (3.32) results in

$$[M_u(x,u,S(x,p,u,t),t) + f_u(x,u,S(x,p,u,t),t)p]\lambda_2' = 0 \qquad (3.36)$$

$$\forall\, t \in [t_o, t_f]$$

which will generally make the leader's problem singular [34]. This is to be expected, because the leader exerts his influence through the time functions resulting from u and u_x, which are actually quite independent, and u_x is not penalized or subjected to any constraint in the initial formulation (2.3)-(3.5). In other words, the leader is more powerful than what a first inspection of the original problem indicates. One way to restrict the leader's strength or to avoid the singular problem could be the inclusion of u_x^i in L, i.e., $L = L(x,u,u_x^1,\ldots,u_x^m,t)$, which would model a self disciplined leader, or to impose a priori bounds on u_x, for example, $\|u_x^i\| \leq K$, $\forall\, t \in [t_o, t_f]$ which could be interpreted as a constitutional restriction on a real life leader.

It could be suggested to the follower to penalize u_x^i in his criterion while u_x^i is not penalized in the leader's criterion. This would lead to the appearance of u_{xx}^i in (3.14) (assuming u_{xx}^i exists). Thus, in addition to (3.36) a similar condition due to u_{xx}^i appears which reinforces the singular character of the problem. If the leader now restricts himself to affine strategies in x, then $u_{xx}^i = 0$ and the resulting optimum is as before. Actually, the leader can restrict himself to a quadratic strategy in x (without affecting his global optimum cost and trajectory) having thus three

influences on the system, namely u, u_x, u_{xx}^i, from which only u is penalized in the leader's criterion. Therefore, the leader will do better. For the follower it is not obvious if he will do better or not.

Let us work out a Linear Quadratic LF game. The leader is penalized for u_x^i as well, by including it in L. We consider the dynamic system

$$\dot{x} = Ax + B_1 u + B_2 v, \quad x(t_o) = x_o, \quad t \in [t_o, t_f] \tag{3.37}$$

and the cost functionals

$$J_1(u,v) = \frac{1}{2}[x_f' K_{1f} x_f + \int_{t_o}^{t_f} (x'Q_1 x + u'R_{11}u + v'R_{12}v + \sum_{i=1}^{m_1} u_x^{i'} R_i u_x^i)dt] \tag{3.38}$$

$$J_2(u,v) = \frac{1}{2}[x_f' K_{2f} x_f + \int_{t_o}^{t_f} (x'Q_2 x + u'R_{21}u + v'R_{22}v)dt] \tag{3.39}$$

where the matrices A, B_i, Q_i, R_{ij}, R_i are continuous functions of time and Q_i, R_{ij}, $R_i > 0$, $R_{11} > 0$ are symmetric. $R_{22} > 0$ is nonsingular $\forall\, t \in [t_o, t_f]$, which guarantees (3.11). The follower's necessary conditions are (recall (3.10)).

$$v = -R_{22}^{-1} B_2' p \tag{3.40}$$

$$\dot{x} = Ax + B_1 u - B_2 R_{22}^{-1} B' p \tag{3.41}$$

$$\dot{p} = -Q_2 x - u_x R_{21} u - A'p - u_x B_1' p \tag{3.42}$$

$$x(t_o) = x_o, \quad p(t_f) = K_{2f} x_f. \tag{3.43}$$

Therefore, the leader's problem is (recall (3.9), (3.14))[5]

---

[5] We assume that Assumption (A) holds. See also Appendix E.

$$\text{minimize } J(u) = \frac{1}{2}[x_f' K_{1f} x_f + \int_{t_o}^{t_f} (x' Q_1 x + u' R_{11} u +$$

$$+ p' B_2 R_{22}^{-1} R_{12} R_{22}^{-1} B_2' p + \sum_{i=1}^{m_1} u_x^{i'} R_i u_x^i)dt] \tag{3.44}$$

subject to:

$$\dot{x} = Ax - B_2 R_{22}^{-1} B_2' p + B_1 u \tag{3.45}$$

$$\dot{p} = -Q_2 x - A' p - u_x B_1' p - u_x R_{21} u \tag{3.46}$$

$$x(t_o) = x_o, \quad p(t_f) = K_{2f} x_f. \tag{3.47}$$

The necessary conditions for the leader in accordance with Theorem 3.2 are (3.45), (3.46), (3.47) and

$$R_{11} u + B_1' \lambda_1 - R_{21}' u_x' \lambda_2 = 0 \tag{3.48}$$

$$[R_1 u_x^1 \vdots \ldots \vdots R_{m_1} u_x^{m_1}] + \lambda_2 (R_{21} u + B_1' p)' = 0 \tag{3.49}$$

$$\dot{\lambda}_1 = -Q_1 x - A' \lambda_1 + Q_2' \lambda_2 \tag{3.50}$$

$$\dot{\lambda}_2 = -B_2 R_{22}^{-1} R_{12} R_{22}^{-1} B_2' p + B_2 R_{22}^{-1} B_2' \lambda_1 + A\lambda_2 + B_1 u_x \lambda_2 \tag{3.51}$$

$$\lambda_1(t_f) = K_{1f} x_f, \quad \lambda_2(t_0) = 0. \tag{3.52}$$

For simplification we assume further that

$$R_i = \gamma_i I, \quad \gamma_i = \gamma > 0, \quad i = 1, \ldots, m_1$$

$$R_{11} = I, \quad R_{22} = I \tag{3.53}$$

and (3.48), (3.49) are easily solved for u and u_x to yield

$$u = -[I + \frac{\|\lambda_2\|^2}{\gamma} R'_{21}R_{21}]^{-1}[B'_1\lambda_1 + \frac{\|\lambda_2\|^2}{\gamma} R'_{21}B'_1p] \tag{3.54}$$

$$u_x = -\frac{1}{\gamma}\lambda_2[p'B_1 + u'R'_{21}] \tag{3.55}$$

which can be substituted into (3.45), (3.46), (3.50), (3.51) to yield a nonlinear system of differential equations, with unknown x, p, λ_1, λ_2 and boundary conditions (3.47) and (3.52). If $\gamma \to +\infty$, then (3.54) and (3.55) yield $u_x \to 0$ and $u \to -B'_1\lambda_1$, and thus the solution tends to the open loop solution, i.e., $u = u(t)$ $v = v(t)$, as the resulting form of (3.45), (3.46), (3.50), (3.51) indicates for $\gamma \to +\infty$ ([27],[28]).

3.5. Relation to the Principle of Optimality

It has been shown in [29] through a counterexample that the Principle of Optimality does not hold for LF games. To make this statement more precise, let us assume that the LF problem of Section 3.2 has been solved in $[t_o, t_f]$ and x^* is the optimal trajectory. While the process is at $(x^*(\tau),\tau)$, where $t_o < \bar{\tau} < t_f$, we stop and solve the same LF game on $[\bar{\tau}, t_f]$ with initial condition $x(\bar{\tau}) = x^*(\bar{\tau})$. Let $\bar{x}^*$ be the optimal trajectory for the second problem. Then $\bar{x}^*$ does not have to coincide with the restriction of x^* on $[\bar{\tau}, t_f]$. The explanation is the following. The leader is faced with the control problem (3.14) which has boundary conditions $x(t_o) = x_o$ and $p(t_f) = \frac{\partial h(x(t_f))}{\partial x}$, given at t_o and t_f. Let (x^*, p^*) be the optimal trajectory of this problem. If the leader is asked to solve the same control problem on $[\bar{\tau}, t_f]$ with boundry conditions $x(\bar{\tau}) = x^*(\bar{\tau})$ and $p(t_f) = \frac{\partial h(x(t_f))}{\partial x}$, there is no necessity for $p(\bar{\tau}) = p^*(\bar{\tau})$!

Even more, if λ_1, λ_2 are the adjoint variables of the leader's control problem on $[t_o, t_f]$ and $\bar\lambda_1$, $\bar\lambda_2$ are the adjoint variables of the leader's control problem on $[\bar t, t_f]$, corresponding to x and p respectively, it will be

$$\lambda_1 (t_f) = \frac{\partial g (x^*(t_f))}{\partial x}, \quad \lambda_2 (t_o) = 0, \quad \bar\lambda_1 (t_f) = \frac{\partial g (\bar x^*(t_f))}{\partial x}, \quad \bar\lambda_2 (\bar t) = 0.$$

If the Principle of Optimality were holding, it should be $\lambda_2 (\bar t) = \bar\lambda_2 (\bar t) = 0$, which is not true. Actually, $\lambda_2 (\bar t) = 0$, $\forall \bar t \in [t_o, t_f]$ is a necessary condition for the Principle of Optimality to hold The condition $\lambda_2 (\bar t) = 0$, $\forall \bar t \in [t_o, t_f]$ can be used for deriving more explicit conditions in terms of the data of the problem for the Principle of Optimality to hold.

Let us consider the linear quadratic game of Section 3.4. As it was shown in the previous paragraph, $\lambda_2 (t) = 0$ $\forall t \in [t_o, t_f]$, is a necessary condition for the Principle of Optimality to hold. With $\lambda_2 \equiv 0$, (3.51) yields

$$-B_2 R_{22}^{-1} R_{12} R_{22}^{-1} B_2' P + B_2 R_{22}^{-1} B_2' \lambda_1 = 0$$

from which, by assuming rank $B_2 = m_2$, we obtain equivalently

$$-R_{12} R_{22}^{-1} B_2' P + B_2' \lambda_1 = 0.$$

Also, (3.49) yields

$$u_x^i = 0, \quad i = 1,\ldots,m_1. \tag{3.56}$$

We conclude that under the assumption rank $B_2 = m_2$, (3.45)-(3.52) simplify to give

$$\dot x = Ax + B_1 u + B_2 v \tag{3.57}$$

$$\dot\lambda_1 = -Q_1 x - A'\lambda_1 \tag{3.58}$$

$$R_{11}u + B_1'\lambda_1 = 0, \quad R_{12}v + B_2'\lambda_1 = 0 \tag{3.59}$$

$$x(t_o) = x_o, \quad \lambda_1(t_f) = K_{1f}x_f \tag{3.60}$$

$$\dot{p} = -Q_2 x - A'p \tag{3.61}$$

$$v = -R_{22}^{-1}B_2'p \tag{3.62}$$

$$p(t_f) = K_{2f}x_f. \tag{3.63}$$

(3.57)-(3.60) show that the leader's problem can be considered as a team problem under the "constraint" (3.56), with optimal solution, say $(u*,v*)$ and (3.61) – (3.63) show that the same $v*$ must be the follower's optimal reaction to the leader's choice u^*. Actually (3.56) is not at all a constraint, since with $\lambda_2 \equiv 0$, (3.46) (where u_x^i appears) is not really considered by the leader. So, the leader operating under (3.45) and wanting to minimize (3.38) may as well choose $u_x^i = 0$, since he is penalized for u_x^i, while u_x^i does not appear in (3.45).

The same analysis and conclusions carry over to the more general game of Section 3.2 (see (3.1)-(3.5) and (3.11)), since the condition $\lambda_2 \equiv 0$ on $[t_o, t_f]$ comes from the demand that the transversality conditions hold $\forall t \in [t_o, t_f]$ and is not affected by the fact that in (3.4) u_x^i is not penalized. Notice that if the leader's cost functional (3.4) is substituted by

$$J_1(u,v) = g(x(t_f)) + \int_{t_o}^{t_f}\{L(x,u,v,t) + \sum_{i=1}^{m_1} u_x^{i'}R_i u_x^i\}dt \tag{3.64}$$

$$R_i > 0, \quad i = 1,\ldots,m_1$$

then (3.56) holds again.

The idea behind the condition $\lambda_2 \equiv 0$ on $[t_o, t_f]$ is that the leader is not really constrained by the follower's adjoint equation and therefore the leader's problem, being independent of the follower's problem, becomes a team control problem.

In conclusion, a necessary condition for the Principle of Optimality to hold for the LF games of Sections 3.2 (and 3.4), is that the leader's problem is actually a team control problem. But for a control problem with fixed initial conditions, the Principle of Optimality does hold. We thus have the "if and only if" statement: The Principle of Optimality holds for the problems of Section 3.2 (and 3.4) (see (3.1)-(3.5), (3.11) and (3.37)-(3.39) respectively) if and only if the leader's problem is a team control problem for both the leader and follower.

CHAPTER 4

SUFFICIENT CONDITIONS FOR LEADER-FOLLOWER AND NASH
STRATEGIES WITH MEMORY

4.1. Introduction

In the present Chapter we consider a continuous time deterministic
differential game with a linear state equation and quadratic cost functionals.
We consider the case where the players have at each instant of time recall of
previous values of the trajectory, i.e. they have memory. Both LF and Nash
equilibrium concepts are considered and sufficient conditions are developed
for a class of problems to have solutions affine in the information. Particular
emphasis is placed on the LF case.

The structure of this Chapter is the following. In 4.2 we give an
example of an LF game where the leader by using previous values of the state
forces the follower to such a reaction that the leader's final cost is the same
as it would have been if both leader and follower were striving to minimize
the leader's cost. The main steps in solving this example serve as an illustra-
tion of how a more general case should be analyzed. In 4.3 we derive sufficiency
conditions for optimality for a control problem of a special form (of interest
on it's own) which are used in the next sections. In 4.4 we apply the results
of 4.3 to a LF game where the leader has recall of the previous trajectory and
the game is such that the solution of the LF game (u^*,v^*) minimizes the leader's
cost over all admissible (u,v), i.e., the leader's problem is actually treated
as a team problem of both the leader and follower. In 4.5 we consider certain
special cases and generalizations of the LF game of 4.4. In 4.6 we apply the
results of 4.3 to a Nash game where the two players have perfect recall of the
whole previous trajectory. One Appendix to this Chapter is provided at the end
of the Thesis.

4.2.   Introductory Example

In this section we provide an example of an LF differential game where the leader uses the previous values of the state in calculating his control values. The game considered is such that the leader, by using this type of strategy forces the follower to such a reaction that the leader's optimal cost is the one he would achieve if both leader and follower had as their common objective the minimization of the leader's cost; i.e., the leader's problem to minimize J_1 is actually treated as a team problem where the team is composed by both the leader and the follower. A similar idea is used implicitely in [44]. The strategies found provide an LF equilibrium pair, with the property above, for any x_o. Also, the dependence of the leader's control values on previous state values is not trivial in the sense that the same result - team solution of the leader's problem - cannot be achieved by strategies depending only on current state value's information. We develop the example in such a way that the proof of the optimality of the indicated strategies is clear. Actually we do not give only one example but provide a way of constructing a whole class of Stackelberg games with the above properties.

Consider the following state equation and cost functionals

$$\dot{x} = 2x + u + v \qquad , \; x(0) = x_o, \;\; t \in [0,1] \qquad (4.1)$$

$$J_1 = 4x(1)^2 + \int_0^1 (6x^2 + u^2 + v^2)dt \qquad (4.2)$$

$$J_2 = 2x(1)^2 + \int_0^1 (q\, x^2 + r v^2)dt \qquad (4.3)$$

where x, u, v are scalar-valued. The solution of the problem

$$\underset{u,v}{\text{minimize }} J_1 \tag{4.4}$$

subject to (4.1)

is

$$\bar{u} = -2kx, \quad \bar{v} = -2kx \tag{4.5}$$

where k solves $-\dot{k} = 3+4k-4k^2$, $k(1)=2$, and is given explicitly by

$$k(t) = \frac{15+e^{8(t-1)}}{10-2e^{8(t-1)}} \tag{4.6}$$

We want to show that there exist q, r, ℓ_1, ℓ_2, so that
the problem

$$\underset{v}{\text{minimize }} J_2$$

$$\text{subject to } \dot{x} = 2x+\ell_1 x+\ell_2 z+v , \quad x(0)=x_o \tag{4.7}$$

$$\dot{z} = x \qquad , \quad z(0)=0$$

has the solution

$$v* = \mu_1 x + \mu_2 z \tag{4.8}$$

and that

$$\ell_1(t)x*(t)+\ell_2(t)z*(t) = -2k(t)x*(t) \quad , \ t\in[0,1] \tag{4.9}$$

$$v*\big|_t = \mu_1(t)x*(t)+\mu_2(t)z*(t) = -2k(t)x*(t), \quad t\in[0,1] \tag{4.10}$$

$$\ell_2(t) \equiv 0 \quad t\in[0,1] \tag{4.11}$$

for any x_o where x* is the common optimal trajectory of the problems (4.7) and
and (4.4) since (4.9) and (4.10) will hold.

It is clear that if conditions (4.9) and (4.10) are satisfied, then the pair $(u = \ell_1(t)x(t) + \ell_2(t)\int_0^t x(\tau)d\tau, \; v = -2k(t)x(t))$ constitutes an LF equilibrium pair for the LF differential game associated with (4.1)– (4.3) and where $U = \{u \mid$ value of u at time t is given by $u(x_t, t)$, where $x_t \in C([0,t], R)$, $x_t(\theta) = x(\theta) \;\; \forall \theta \in [0,t]$, $u(x_t, t)$ is Frechet differentiable in x_t and piecewise continuous in $t \in [0,1]\}$, $V = \{v \mid v$ is a function of $x(t)$ and t, at time t, $v(x,t)$ is continuous in $x \in R$ and piecewise continuous in $t \in [0,1]\}$.

We set

$$\alpha(t) = 2 - 4k(t) \tag{4.12}$$

and thus the optimal trajectory for the problem (4.4) is

$$x^*(t) = e^{\int_0^t \alpha(\tau)d\tau} \cdot x_0 \tag{4.13}$$

$$z^*(t) = \int_0^t [e^{\int_0^\tau \alpha(\sigma)d\sigma}]d\tau \cdot x_o \tag{4.13}$$

The solution v^* of (4.7) is

$$v^* = -\frac{1}{r}(p_1 x + p_2 z) \tag{4.14}$$

where

$$-\dot{p}_1 = 2(2 + \ell_1)p_1 + 2p_2 + q - \frac{1}{r}p_1^2 \qquad, \; p_1(1) = 2$$

$$-\dot{p}_2 = \ell_2 p_1 + (2 + \ell_1)p_2 + p_3 - \frac{1}{r}p_1 p_2 \qquad, \; p_2(1) = 0 \tag{4.15}$$

$$-\dot{p}_3 = 2\ell_2 p_2 - \frac{1}{r}p_2^2 \qquad, \; p_3(1) = 0.$$

Substituting (4.13) and (4.14) in (4.9), (4.10) we obtain

$$P_1 = 2r \; k - p_2 \varphi$$

$$\ell_1 = -2k - \ell_2 \varphi \tag{4.16}$$

where

$$\varphi(t) = (\int_0^t e^{\int_0^\tau \alpha(\sigma)d\sigma} \, d\tau)(e^{\int_0^t \alpha(\tau)d\tau}).$$ (4.17)

(Since $\varphi = z/x$, it is easy to see that $\dot\varphi = 2-(2-4k)\varphi$.) Substituting p_1, ℓ_1 from (4.16) into (4.15) we obtain further

$$2\ell_2 \varphi rk - (p_2 + \varphi p_3) - q + 4rk^2 + 6r - 2\dot r k = 0$$ (4.18)

$$-\dot p_2 = \ell_2(2rk-p_2\varphi) - (2-2k-\ell_2\varphi)p_2 + p_3 - \frac{1}{r}(2rk-p_2\varphi)p_2$$ (4.19)

$$-\dot p_3 = 2\ell_2 p_2 - \frac{1}{r} p_2^2$$ (4.20)

$$2r(1)k(1) - p_2(1)\varphi(1) = 2$$ (4.21)

$$p_2(1) = 0$$ (4.22)

$$p_3(1) = 0.$$ (4.23)

From (4.19) and (4.20), setting $w \overset{\Delta}{=} p_2 + \varphi p_3$ we obtain

$$\dot w = -2rk\ell_2 - (2-4k)w, \qquad w(1) = 0.$$ (4.24)

Solving (4.18) for $p_2+\varphi p_3$ and substituting in (4.24) we obtain finally the following system equivalent to (4.18)-(4.23)

$$\dot\ell_2 \varphi rk + \ell_2[rk + 2(1-2k)\varphi rk + \frac{d}{dt}(\varphi rk)] +$$
$$+ [\frac{d}{dt}(2rk^2 + 3r - \dot r k) + 2(1-2k)(2rk^2 + 6r - 2\dot r k)] + [(1-2k)q - \frac{1}{2}\dot q] = 0$$ (4.25)

$$-\dot p_3 = 2\ell_2(w-\varphi p_3) - \frac{1}{r}(w-\varphi p_3)^2$$ (4.26)

$$w = 2\ell_2 \varphi rk - q + 4rk^2 + 6r - 2\dot r k$$ (4.27)

$$p_2 = w - \varphi p_3$$ (4.28)

$$\ell_2(1) = \frac{1}{2}(-11 + 4\dot r(1) + q(1))\varphi(1)^{-1}$$ (4.29)

$$p_3(1) = 0$$ (4.30)

$$r(1) = \frac{1}{2}.$$ (4.31)

We can choose now r, q, ℓ_2, ℓ_1 as to satisfy (4.25)-(4.31) and (4.16). We choose r(t) to be a twice differentiable function of $t \in [0,1]$ with $r(t) > 0$, $r(1) = \frac{1}{2}$ and q(t) to be a differentiable function of t. Obviously q and r can be chosen so that the linear differential equation for ℓ_2 (4.25) with initial condition (4.29) has the solution $\ell_2(t) \neq 0$. For example let $r \equiv \frac{1}{2}$, $q \equiv$ constant $\neq 11$. Notice that the differential equation (4.25) for ℓ_2 can be solved explicitly for ℓ_2 as soon as r and q are specified since φ and k are known. Nonetheless since $\varphi(0) = 0$, the point $t = 0$ is a singular point of this differential equation. The singularity was sort of expected to appear since as it has been shown in Chapter 3, the leader's problem is singular with respect to the partial derivative $\frac{\partial(u(x(t),t)}{\partial x}$ of his control and arguments similar to those in Chapter 3 can be used to show that this holds even for the case where u is allowed to be of the more general form $u(x_t, t)$. Notice also that the only essential restriction on the follower's cost, in order for the leader to achieve his team solution (allowing even $\ell_2 \equiv 0$) is that $r(1) = \frac{1}{2}$.

If the leader were allowed to use a strategy u(x,t) which is perhaps nonlinear in the current state x(t), but he were not permitted to use previous values of the state, then it should again be true that

$u(x^*(t),t) = -2k(t)x^*(t)$ for every x_o, i.e.

$$u(e^{\int_0^t a(\tau)d\tau} x_o, t) = -2k(t)e^{\int_0^t a(\tau)d\tau} x_o \qquad \forall x_o \in R$$

from which we obtain that u is linear in x. Therefore, we conclude that for the given example, if the leader wishes to achieve his team solution (for any x_o) when he applies his strategy and cannot do that with a linear strategy in x(t), he cannot do it with a nonlinear in x(t) strategy either. Therefore, use of memory is his only way to achieve his team solution.
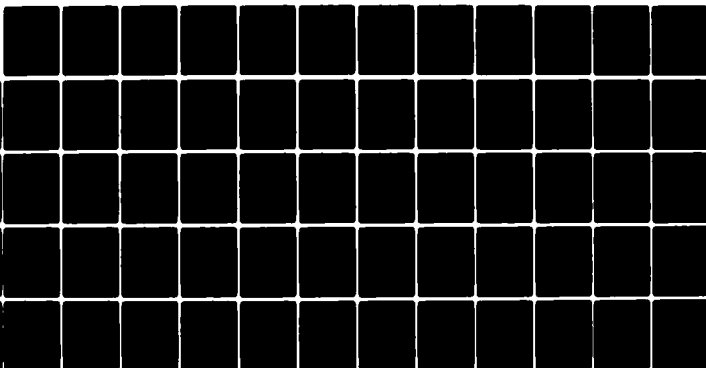
In the example presented here the two crucial steps were the identifications (4.9), (4.10) and the use of the fact that the conditions (4.14), (4.15) are sufficient to characterize completely the optimal reaction of the follower to the leader's strategy $u = \ell_1(t)x(t) + \ell_2(t)\int_0^t x(\tau)d\tau$. Therefore, in order to generalize the procedure presented to cases where more general types of strategies are used by the leader, one should provide sufficient conditions for the problem faced by the follower, in addition to imposing identifications similar to (4.9) and (4.10). In the next section we prove sufficiency conditions for a special type of control problem, which we will use later in guaranteeing the optimality of the follower's reaction, when the leader uses strategies represented as continuous linear functionals over the whole previous trajectory.

4.3. A Control Problem With State-Control Constraints.

Consider the problem (P):

$$\text{minimize } \overline{J} = \frac{1}{2}\left[x'(t_f)Fx(t_f) + \int_{t_0}^{t_f}(x'(t)Q(t)x(t) + u'(t)R(t)u(t))dt\right] \tag{4.32}$$

subject to: $\dot{x}(t) = A(t)x(t) + B(t)u(t), \; x(t_0) = x_0$ \hfill (4.33)

$$\int_{t_0}^{t_f}[d_s\eta(t,s)]x(s) + \int_{t_0}^{t_f}[d_s\eta_1(t,s)]u(s) = q(t) \tag{4.34}$$

$$u \in L_{\infty,m}$$

where the matrices A, B, Q = Q' $\geq$ 0, R = R' $\geq$ 0, are piecewise continuous functions of time, $x(t) \in R^n$, $u(t) \in R^m$, and the interval $[t_0, t_f]$ the matrix

$F = F' \geq 0$ and $q \in L_{1,k}$ are fixed. The solution $x(t)$ of (4.33) is assumed to be absolutely continuous, so that (4.33) holds almost everywhere with respect to the Lebesgue measure in $[t_0, t_f]$. The integrals in (4.34) should be interpreted as Lebesgue-Stieltjes integrals. The matrix valued function $\eta(t,\theta)$, $\eta: [t_0, t_f] \times R \to R^{k \times n}$ is measurable in (t,θ), normalized so that

$$\eta(t,\theta) = \begin{cases} 0 & , \text{ for } \theta \geq t_f \\ \eta(t,t_0), & \text{ for } \theta \leq t_0 . \end{cases} \tag{4.35}$$

$\eta(t,\theta)$ is continuous from the left in θ on (t_0, t_f), $\eta(t,\theta)$ has bounded variation in θ on $[t_0, t_f]$ for each t, and there is a $c \in L_{1,1}$ such that

$$\left\| \int_{t_0}^{t_f} [d_s \eta(t,s)] \varphi(s) \right\|_{L_1} \leq c(t) \| \varphi \|_C \tag{4.36}$$

for all $t \in [t_0, t_f]$ and for all $\varphi \in C_n$. Exactly the same assumptions hold for $\eta_1: [t_0, t_f] \times R \to R^{k \times m}$ with c_1 replacing c in (4.36). η and η_1 are given for the problem (P). The dimension k is arbitrary but fixed.

Problem (P) is of interest to us since we will use the results of this section in the next ones where we will consider games with delayed information structure. Nonetheless it is of interest on it's own. It is worthy to point out that (P) is of a quite general form, since for example, the problem (P')

$$\text{minimize } \frac{1}{2} \left[x'(t_f) F x(t_f) + \int_{t_0}^{t_f} (y'(t) Q y(t) + u_1'(t) R(t) u_1(t)) dt \right]$$

$$\text{subject to: } \dot{x}(t) = \int_{t_0}^{t_f} [d_s \eta^1(t,s)] x(s) + \int_{t_0}^{t_f} [d_s \eta^2(t,s)] \bar{u}(s)$$

$$y(t) = \int_{t_o}^{t_f} [d_s \eta^3(t,s)] x(s) \tag{4.37}$$

$$u_1(t) = \int_{t_o}^{t_f} [d_s \eta^4(t,s)] \bar{u}(s)$$

$$x(t_o) = x_o$$

can be brought to the form (P) by introducing

$$u_2(t) = \int_{t_o}^{t_f} [d_s \eta^1(t,s)] x(s)$$

$$u_3(t) = \int_{t_o}^{t_f} [d_s \eta^2(t,s)] x(s) \tag{4.38}$$

$$u_4(t) = y(t).$$

Using (4.37), (4.38), (P') can be written equivalently

$$\text{minimize } \frac{1}{2} [x'(t_f) \, Fx(t_f) + \int_{t_o}^{t_f} (u_4' Q u_4 + u_1' R u_1) dt]$$

$$\tag{4.39}$$

subject to $\dot{x}(t) = u_2(t) + u_3(t)$

$$u_1 = \int_{t_o}^{t_f} [d\eta^4] \bar{u}, \qquad u_2 = \int_{t_o}^{t_f} [d\eta^1] x, \qquad u_3 = \int_{t_o}^{t_f} [d\eta^2] x, \qquad u_4 = \int_{t_o}^{t_f} [d\eta^3] x \tag{4.40}$$

where the role of x and u in (4.33), (4.34) is played now by x and

$(\bar{u}, u_1, u_2, u_3, u_4)$ respectively. Clearly (4.40) is of the form (4.34).

In the following theorem we give sufficiency conditions for optimality for the problem (P). The proof is carried out by reformulating (P) as a constrained optimization problem in a Banach space and is given in Appendix G.

Theorem 4.1:Consider the problem (P) and assume that there exist functions

$\mu: [t_0, t_f] \to R^n$, $\lambda \in L_{\infty, k}$, $x^*: [t_0, t_f] \to R^n$, $u^* \in L_{\infty, m}$, where μ is of bounded variation on $[t_0, t_f]$ and continuous from the right on (t_0, t_f), and x^* is absolutely continuous, which satisfy (4.33), (4.34) and

$$- \int_t^{t_f} (R(\tau)u^*(\tau) + B'(\tau)\mu(\tau))d\tau + \int_{t_0}^{t_f} \eta_1'(\tau, t)\lambda(\tau)d\tau = 0 \qquad (4.41)$$

$$\mu(t) - \int_t^{t_f} (Q(\tau)x^*(\tau) + A'(\tau)\mu(\tau))d\tau + \int_{t_0}^{t_f} \eta'(\tau, t)\lambda(\tau)d\tau = Fx(t_f) \qquad (4.42)$$

Then u^*, x^* solve (P).

It is easy to see that in case η, $\eta_1 \equiv 0$, then (4.41) and (4.42) reduce to

$$R(t)u^*(t) + B'(t)\mu(t) = 0$$

$$-\dot{\mu}(t) = Q(t)x(t) + A'(t)\mu(t), \qquad \mu(t_f) = Fx(t_f)$$

as it should be expected.

Theorem 4.1 can be easily extended to the case where cross terms u'Lx exist in the integrand of (4.32) and to cases where more general convex cost functionals (4.32) are considered.

4.4. A Leader Follower Game with Memory

Consider the dynamic system

$$\dot{x}(t) = Ax(t) + B_1\bar{u}(t) + B_2\bar{v}(t), \qquad x(t_o) = x_o, \qquad t \in [t_o, t_f] \qquad (4.43)$$

and the cost functionals

$$J_1 = \frac{1}{2} [x'(t_f) \ F_1 x(t_f) + \int_{t_o}^{t_f} (x'(t) Q_1 x(t) + \bar{u}'(t) R_{11} \bar{u}(t) + \bar{v}'(t) \ R_{12} \bar{v}(t)) dt] \tag{4.44}$$

$$J_2 = \frac{1}{2} [x'(t_f) \ F_2 x(t_f) + \int_{t_o}^{t_f} (x'(t) Q_2 x(t) + \bar{u}'(t) R_{21} \bar{u}(t) + \bar{v}(t)' R_{22} \bar{v}(t)) dt] \tag{4.45}$$

where the matrices A, B_i, $Q_i = Q_i' \geq 0$, $R_{ij} = R_{ij}' \geq 0$ are piecewise continuous functions of time over $[t_o, t_f]$ and R_{11}, R_{22}, R_{12} are nonsingular $\forall t \in [t_o, t_f]$. The matrices $F_i = F_i' \geq 0$ and the time interval $[t_0, t_f]$ are fixed.

Consider the Stackelberg game associated with (4.43)-(4.45). The admissible strategies of the leader are of the form

$$u(x_t, t) = \int_{t_o}^{t} [d_s \eta(t,s)] x(s) \tag{4.46}$$

where η is as in (4.35)-(4.36), so that $u(\cdot, t)$ is a continuous linear functional on $C([t_o, t], R^{m_1})$ for each $t \in [t_o, t_f]$. The admissible strategies of the follower are of the form $v(x,t)$, $x \in R^n$, $t \in R$, where v is continuously differentiable in x and piecewise continuous in t. All the matrices in (4.43)-(4.45) are considered to be of appropriate dimensions. By x_t, $\bar{u}$, $\bar{v}$ we mean

$$x_t : [t_o, t] \to R^n, \ x_t(\theta) = x(\theta), \quad \forall \theta, t \in [t_o, t_f] \tag{4.47}$$

$$\bar{u}(t) = u(x_t, t), \quad \bar{v}(t) = v(x(t), t) \tag{4.48}$$

where x(t) is the trajectory of (4.43) for given u and v. For each choice of u and v the behavior of the dynamic system (4.43) and the calues of J_1, J_2 are unambiguously defined, assuming that the solution of (4.43) exists over $[t_o, t_f]$. Actually when the strategy (4.46) is considered one might w.l.o.g. restrict η to be 0 for $s \geq t$, $t \in [t_o, t_f]$. The costs of the leader (J_1) and of the follower (J_2) are functions of u and v. We denote by U and V the

sets of the admissible strategies for the leader and follower respectively. With these explanations the LF game associated with (4.43)-(4.45) is clearly defined.

In the sequel we single out a subclass of LF games with the nice property that the leader achieves the best possible outcome for himself; i.e., the leader's and follower's strategies constitute together an optimal control law for the control problem with cost functional $J_1(u,v)$ subject to the constraints of the state equation. The procedure followed is the following: First solve the leader's problem as a control problem with controls u,v. Let $(\bar{u}*(t),\bar{v}*(t)),x*(t)$ be the optimum control pair and trajectory, where $\bar{u}*(t),\bar{v}*(t)$ are piecewise continuous functions of time. Consider any function $\tilde{u} \in U$ such that $\tilde{u}(x_t^*,t) = \bar{u}*(t)$ $\forall t \in [t_0,t_f]$. Second, solve the following inverse control problem: with $u = \tilde{u}$ in the follower's cost, and the state equation, minimize $J_2(\tilde{u},v)$ and seek conditions so that $v*$ solves this problem and the resulting optimal trajectory for this problem is again $x*(t)$. So, if these conditions are assumed to hold apriori, then the pair $(\tilde{u},\bar{v}*)$ constitutes a Stackelberg pair. One may derive conditions, by solving the inverse control problem, where $\tilde{u}$ depends only on $x(t)$, or on almost any subset of $\{x(\tau);t_0 < \tau \leq t\}$ for each t. One may also single out a whole class of LF problems where the inverse control problem does not have $v*$ as its solution, whatever is the $\tilde{u}$. For example if $J_2 = \int_{t_0}^{t_f} \bar{v}'(t)\,\bar{v}(t)dt$, then $v*$ will be optimum if and only if $\bar{v}*(t) \equiv 0$. It is trivial to exhibit now a class of J_1's and A, B_1, B_2, so that $\bar{v}*(t) \neq 0$.

Consider the control problem

minimize J_1

(4.49)

subject to $\bar{u}, \bar{v}$ piecewise continuous functions of t

and (4.43)

Then (4.49) has the solution

$$\bar{u}*(t) = -R_{11}^{-1}B_1'Kx(t), \bar{v}*(t) = -R_{12}^{-1}B_2'Kx(t) \qquad (4.50)$$

where K is the continuous solution of

$$-\dot{K} = KA + A'K + Q_1 - K[B_1R_{11}^{-1}B_1' + B_2R_{12}^{-1}B_2']K, K(t_f) = F_1, \quad t\in[t_o, t_f] \qquad (4.51)$$

which is assumed to exist. Let $\Phi(t, t_o)$ be the transition matrix of the resulting closed loop system in (4.43), i.e.

$$\frac{\partial\Phi(t, t_o)}{\partial t} = (A - B_1R_{11}^{-1}B_1'K - B_2R_{12}^{-1}B_2'K)\Phi(t, t_o), \ \Phi(t_o, t_o) = I, \quad t\in[t_o, t_f]. \qquad (4.52)$$

Then the optimal trajectory x* and control values of $\bar{u}*, \bar{v}*$ for (4.49) are given by

$$x*(t; t_o, x_o) = \Phi(t, t_o)x_o \qquad (4.53)$$

$$\bar{u}*(t) = -R_{11}^{-1}B_1'K\Phi(t, t_o)x_o \qquad (4.54)$$

$$\bar{v}*(t) = -R_{12}^{-1}B_2'K\Phi(t, t_o)x_o. \qquad (4.55)$$

Let η be as in (4.35), (4.36) with $n(t, \theta) = 0$ for $\theta \geq t$ and satisfy the identity

$$\int_{t_o}^{t} [d_s\eta(t, s)]\Phi(s, t) \equiv -R_{11}^{-1}(t)B_1'(t)K(t), \ t\in[t_o, t_f]. \qquad (4.56)$$

If η satisfies (4.56), then

$$\bar{u}*(t) = \int_{t_o}^{t} [d_s\eta(t, s)]\Phi(s, t_o)x_o = \int_{t_o}^{t} [d_s\eta(t, s)]x*(s). \qquad (4.57)$$

(4.56) characterizes all the η's which result in the same $\bar{u}*(t)$ (4.54), i.e. provides a class of different representations of $\bar{u}*(t)$ as a linear continuous functional of $x_t^* = \{x*(\theta); t_o \leq \theta \leq t\}$. This class of η's is not empty, since for example

$$\bar{\eta}(t,\theta) = \begin{cases} 0 & , \text{ for } \theta \geq t, \quad t \in (t_o, t_f] \\ -R_{11}^{-1}(t)B_1'(t)K(t), & \text{ for } \theta < t, \quad t \in (t_o, t_f] \text{ and} \\ & \text{ for } \theta \leq t_o, \quad t = t_o \end{cases} \qquad (4.58)$$

satisfies (4.56). For fixed t, the set of all $\eta(t,\cdot)$ which satisfy (4.56) is the hyperplane $H_t = \{\eta(t,\cdot) \mid \eta(t,\cdot) \in NBV([t_o,t], R^{m_1 \times n}), \eta(t,\cdot)$ perpendicular to $\Phi(\cdot,t)\}$ shifted by $\bar{\eta}(t,\cdot)$ from the origin in the dual space of $C([t_o,t], R^{n \times n})$. A useful class of η's which satisfy (4.56) is given by

$$\eta(t,s) = \bar{\eta}(t,s) + H_o(t,s) + \sum_{i=1}^{p} A_i(t,s)d(s - \rho_i(t)) \qquad (4.59)$$

where H_o is absolutely continuous in s for each t, $A_i : [t_o,t_f] \times R \to R^{m_1 \times n}$ continuous, $\rho_i : [t_o,t_f] \to R$ continuous, $d(s) = 0$ for $s \leq 0$ and $d(s) = 1$ for $s > 0$, and

$$\int_{t_o}^{t} \frac{\partial H_o(t,s)}{\partial s} \Phi(s,t)ds + \sum_{i=1}^{p} A_i(t,\rho_i(t))\Phi(\rho_i(t),t) \equiv 0 \qquad \text{on } [t_o,t_f]. \quad (4.60)$$

Another η which satisfies (4.56) is

$$\tilde{\eta}(t,s) = \begin{cases} 0, & \text{for } \theta \geq t/2, \quad t \in (t_o,t_f] \\ 0, & \text{for } \theta > t_o, \quad t = t_o \\ -R_{11}^{-1}(t)B_1'(t)K(t)\int_0^s \Phi(t,\sigma)d\sigma \cdot \dfrac{2}{t-t_o}, \\ \qquad \text{for } \theta < t/2, \quad t \in (t_o,t_f] \\ -R_{11}^{-1}(t_o)B_1'(t_o)K(t_o), & \text{for } \theta \leq t_o, \quad t = t_o. \end{cases} \qquad (4.61)$$

Notice that

$$\bar{u}^*(t) = \int_{t_o}^{t_o + \frac{t-t_o}{2}} [d_s \widetilde{\eta}(t,s)] x^*(s) \tag{4.62}$$

i.e. only the first half of the trajectory up to time t is used in calculating $\bar{u}^*(t)$.

<u>Theorem 4.2</u>: Assume that there exists a function η^* as in (4.35),(4.36) with $\eta^*(t,\theta) = 0$ for $\theta \geq t$ and an n×n matrix function $P : [t_o, t_f] \to R^{n \times n}$ which satisfy

$$\int_{t_o}^{t} [d_s \eta^*(t,s)] \Phi(s,t) = -R_{11}^{-1}(t) B_1'(t) K(t), \qquad t \in [t_o, t_f] \tag{4.63}$$

$$R_{22}^{-1}(t) B_2'(t) P(t) = R_{12}^{-1}(t) B_2'(t) K(t), \qquad t \in [t_o, t_f] \tag{4.64}$$

$$P(t) + \int_{t}^{t_f} \{-A'(\tau) P(\tau) - Q_2(\tau) + \eta^{*'}(\tau,t) B_1'(\tau) P(\tau)$$

$$+ \dot{\eta}^{*'}(\tau,t) R_{21}(\tau) R_{11}^{-1}(\tau) B_1'(\tau) K(\tau)\} \Phi(\tau,t) d\tau = F_2 \Phi(t_f,t), \qquad t \in [t_o,t_f] \tag{4.65}$$

Then the pair

$$u^*(x_t,t) = \int_{t_o}^{t} [d_s \eta^*(t,s)] x(s) \tag{4.66}$$

$$v^*(x(t),t) = -R_{12}^{-1} B_1'(t) K(t) x(t) \tag{4.67}$$

constitutes an equilibrium pair for the LF game associated with (4.43)-(4.45) for any x_o with strategy spaces U and V.

<u>Proof</u>: We set

$$\bar{\lambda}(t) = P(t) \Phi(t,t_o) x_o. \tag{4.68}$$

Then the vector $\lambda(t) = (-1, \bar{\lambda}(t)')'$ and the control

$$v = -R_{22}^{-1} B_2' \bar{\lambda}(t) \tag{4.69}$$

satisfy the sufficiency conditions of Theorem 4.1 for the problem

$$\text{minimize } J_2(u*(x_t,t),v)$$

$$\text{subject to } v \in V \text{ and } (4.43) \tag{4.70}$$

where u is kept fixed equal to u*. That the u* in (4.66) is the leader's best reaction to v* in (4.47) is an immediate consequence of the fact that the pair (4.66), (4.67) solves the problem (4.49). □

The case where the leader's strategy is allowed to be of the form

$$\int_{t_0}^{t} [d_s \eta(t,s)] y(t,s)$$

where $y(t,x) = C(t,x)x(s)$, $\frac{d}{d_s} C(t,s) = 0$, a.e. $t_0 \leq s \leq t \leq t_f$ with $\eta(t,s) \cdot C(t,s)$ as in (4.35)-(4.36) can also be considered. The property $\frac{d}{d_s} C(t,s) = 0$, a.e. $t_0 \leq s \leq t \leq t_f$ allows one to write $\int_{t_0}^{t} [d_s \eta(t,s)] y(t,s) = \int_{t_0}^{t} d_s [\eta(t,s)C(t,s)] x(s)$ and thus to use directly Theorem 4.1. We only mention that in this case the leader has restricted memory and $\eta^* \cdot C$ should play the role of T^* in (4.63)-(4.67) in the corresponding sufficiency conditions.

For given η^*, (4.65) is an integral equation for P(t). Since it has a Volterra kernel, if in addition it holds that $A'(\tau) - \eta^{*'}(\tau,t)'B_1(\tau)$ is bounded by some M for any $t_0 \leq \tau \leq t$, $t_0 \leq t \leq t_f$, then the Neumann series for (4.65) is always uniformly convergent and furnishes the unique solution of (4.65), see [48]. If $\eta^*(t,s)$ is of the form

$$\eta^*(t,s) = \sum_{i=1}^{k} H_i'(t) \cdot H^{i'}(s), \quad H_i(t) \in R^{pxm_1}, \quad H^i(s) \in R^{nxp} \tag{4.71}$$

then (4.65) can be written as

$$P(t) + \int_{t}^{t_f} \{-A'(\tau) + \Sigma H^i(t)H_i(\tau)B_1'(\tau)\} P(\tau)\Phi(\tau,t)d\tau = F_2\Phi(t_f,t) +$$

$$+ \int_{t}^{t_f} \{Q_2(\tau) - \eta^*(\tau,t)R_{21}(\tau)R_{11}^{-1}(\tau)B_1'(\tau)K(\tau)\}\Phi(\tau,t)d\tau \tag{4.72}$$

which is an integral equation for P with a kernel of finite rank and thus its solution is of the form

$$P(t) = \Xi_0 \Phi(t_o, t) + \sum_{i=1}^{k} H^i(t) \Xi_i \Phi(t_o, t) \qquad (4.73)$$

where $\Xi_0, \Xi_1, \ldots, \Xi_k$ are constant matrices which can be found as solutions of algebraic linear equations. In this case, checking (4.64) is easy as soon as the Ξ_i's in (4.73) are found.

If $(B_2' K B_2)^{-1}$ exists over $[t_o, t_f]$, (it suffices that rank $B_2 = m_2$ and $F_1 > 0$), then (4.64) is equivalent to

$$P(t) = M(t) + Y(t), \quad M(t) = K B_2 (B_2' K B_2)^{-1} R_{22} R_{12}^{-1} B_2' K \qquad (4.74)$$

$$B_2'(t) Y(t) \equiv 0 \quad \text{on } [t_o, t_f] \qquad (4.75)$$

and (4.65) can be transformed into an integral equation for Y.

Theorem 4.2 suggests that for an LF game with given A, B_i, Q_i, R_{ij}, F_i, one may try to find η^* and P which satisfy (4.63)-(4.65) and then consider (4.66), (4.67) as a solution. Also, by solving (4.65) for Q_2, one can exhibit a whole class of LF games with solution (4.66), (4.67), where η^*, P, K, A, B_1, B_2, F_1, R_{11}, R_{12}, R_{22} are chosen as to satisfy (4.51), (4.52), (4.63), (4.64), $F_2 = P(t_f)$ and R_{21} is chosen arbitrarily.

4.5. Special Cases and Generalizations

We first apply the results of Theorem 4.2 to two special cases.

Case i: Let $\eta^* = \bar{\eta}$ as in (4.58). Then u* in (4.66) assumes the form

$$u^*(x_t, t) = -R_{11}^{-1} B_1' K x(t).$$

(4.63) is satisfied and (4.65) simplifies to

$$-\dot{P}(t) = P(A-B_1R_{11}^{-1}B_1'K) + (A-B_1R_{11}^{-1}B_1'K)'P + Q_2$$
$$+ KB_1R_{11}^{-1}R_{12}R_{11}^{-1}B_1'K - PB_2R_{22}^{-1}B_2'P, \quad P(t_f) = F_2 \quad (4.76)$$

If $(B_2'KB_2)^{-1}$ exists and is differentiable on $[t_o, t_f]$ and B_2, $KR_{22}R_{12}^{-1}B_2'K$ are differentiable on $[t_o, t_f]$ and of constant rank, then all the R_{22}, Q_2, F_2, P with $R_{22} > 0$, $P > 0$ which satisfy (4.64) and (4.76) are given by (see [43])

$$R_{22} = V\Gamma V', \quad (4.77)$$

$$P = M + Y \quad (4.78)$$

$$Q_2 = -\dot{P} - P(A-B_1R_{11}^{-1}B_1'K) - (A-B_1R_{11}^{-1}B_1'K)'P \quad (4.79)$$

$$- KB_1R_{11}^{-1}R_{12}R_{11}^{-1}B_1'K + PB_2R_{22}^{-1}B_2'P \quad (4.80)$$

$$F_2 = Q_2(t_f) \quad (4.81)$$

where

$$B_2'KB_2 = V\Lambda V^{-1}, \quad \Lambda = \text{Jordan diagonal form} \quad (4.82)$$

$$\Gamma\Lambda = \Lambda\Gamma, \quad \Gamma = \Gamma' > 0 \quad (4.83)$$

$$B'Y = 0, \quad Y = Y' \geq 0. \quad (4.84)$$

If Γ and Y do not satisfy $\Gamma > 0$, $Y \geq 0$ then one cannot conclude that $R_{22} > 0$ and $P \geq 0$ respectively. Y and R_{12} have to be chosen properly differentiable so that P exists and is piecewise continuous. The

above construction does not guarantee $Q_2 \geq 0$, $F_2 \geq 0$.

<u>Case ii</u>: Let $\eta^* = \eta_1 + \eta_2$ where

$$\eta_1(t,s) = \begin{cases} -R_{11}^{-1}(t)B_1'(t)L_1(t), & \text{for } s < t \quad t \in (t_0, t_f]. \\[2ex] -R_{11}^{-1}(t_0)B_1'(t_0)L_1(t_0), & \text{for } s \le t_0, \\[2ex] 0, & \begin{array}{l} \text{for } s \ge t, \quad t \in (t_0, t_f] \text{ and} \\ \text{for } s > t_0, \quad t = t_0 \end{array} \end{cases}$$

$$\eta_2(t,s) = \begin{cases} (s-t)L_2(t), & \text{for } s < t, \quad t \in [t_0, t_f] \\[2ex] 0, & \text{for } s \ge t \end{cases} \tag{4.85}$$

where L_1, L_2 are real valued matrices. Then u^* in (4.66) assumes the form

$$u^*(x_t, t) = -R_{11}^{-1}(t)B_1'(t)L_1(t)x(t) + L_2(t)\int_{t_0}^t x(s)ds \tag{4.86}$$

and (4.63), (4.64), (4.65) simplify to

$$-R_{11}^{-1}(t)B_1'(t)L_1(t) + L_2(t)\int_{t_0}^t \Phi(s,t)ds = -R_{11}^{-1}(t)B_1(t)K(t) \tag{4.87}$$

$$R_{22}^{-1}(t)B_2'(t)P(t) = -R_{12}^{-1}(t)B_2'(t)K(t) \tag{4.88}$$

$$P(t) + \int_t^{t_f} \{ -A'(\tau)P(\tau) - Q_2(\tau) - L_1'(\tau)B_1(\tau)R_{11}^{-1}(\tau)B_1'(\tau)P(\tau) +$$

$$+ (t-\tau)L_2'(\tau)B_1'(\tau)P(\tau) - L_1'(\tau)B_1(\tau)R_{11}^{-1}(\tau)R_{21}(\tau)R_{11}^{-1}(\tau)B_1'(\tau)K(\tau) +$$

$$+ (t-\tau) L_2'(\tau)R_{21}(\tau)R_{11}^{-1}(\tau)B_1'(\tau)K(\tau) \} \Phi(\tau,t)d\tau = F_2\Phi(t_f,t). \tag{4.89}$$

The cases i and ii were special cases of the case considered in the previous section. We will consider now cases where the leader uses the previous strategy values as well. In the LF game considered in Section 4.4 , the value of the leader's strategy at time t was allowed to depend on the previous trajectory $x_t = \{x(\theta); t_0 \le \theta \le t\}$. More generally one may allow that the values $\bar{u}(t)$ of the admissible strategies of u of the leader at time t depend not only on the previous values of x but also on those of v. Assuming this dependence to be linear we have

$$\overline{u}(t) = \int_{t_o}^{t} [d_s \eta_1(t,s)] x(s) + \int_{t_o}^{t} [d_s \eta_3(t,s)] v(s)$$

or more generally

$$q(t) = \int_{t_o}^{t} [d_s \eta_1(t,s)] x(s) + \int_{t_o}^{t} [d_s \eta_2(t,s)] \overline{u}(s) + \int_{t_o}^{t} [d_s \eta_3(t,s)] \overline{v}(s) \quad (4.90)$$

$q \in L_{1,k}$ fixed.[4] (The η_1, η_2, η_3 in (4.90) are as in (4.35), (4.36).) So, for a given choice η_1, η_2, η_3 by the leader, the follower is faced with the problem

$$\text{minimize } \frac{1}{2} [x'(t_f) F_2 x(t_f) + \int_{t_o}^{t_f} (x'(t) Q_2 x(t) + \overline{u}'(t) R_{21} \overline{u}(t) + \overline{v}(t) R_{22} \overline{v}(t)) dt]$$

$$\text{subject to } \dot{x}(t) = Ax(t) + B_1 \overline{u}(t) + B_2 \overline{v}(t), \quad x(t_o) = x_o \quad (4.91)$$

(4.90) and $\overline{u}, \overline{v}$ piecewise continuous functions of time.

Theorem 4.1 can be now used to derive sufficeint conditions for problem (4.91).

A simple version of (4.90) is

$$\overline{u}(t) = \int_{t_o}^{t} [d_s \overline{\eta}_1(t,s)] x(s) + \int_{t_o}^{t} [d_s \overline{\eta}_2(t,s)] z(s) + L(t) \overline{v}(t) \quad (4.92)$$

where

$$\dot{z}(t) = \overline{A}_1(t) x(t) + \overline{A}_2(t) z(t) + \overline{B}_1(t) \overline{u}(t) + \overline{B}_2(t) \overline{v}(t), \quad z(t_o) = z_0, \quad (4.93)$$

and the matrices $L, \overline{A}_i, \overline{B}_i$ are real valued piecewise continuous functions of time and $z(t) \in R^{\ell}$, ℓ arbitrary. For the linear system (4.43) with quadratic costs (4.49), (4.45), we augment (4.93) to (4.43), set $x = (x'z')'$ and the system is

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[4] Notice that in (4.90), $\overline{u}(t)$ depends on its own previous values. If $\overline{u}(t)$ were allowed to be any function of $x(\theta), \overline{v}(\theta), t_o \leq \theta \leq t$, then the dependence of $\overline{u}(t)$ on it's previous values would not buy the leader anything additional. But if $\overline{u}(t)$ is restricted to depend on $x(\theta), \overline{v}(\theta), t_o \leq \theta \leq t$ in a special form (like in (4.90), see also (4.92)-(4.96)), then allowing dependence of $\overline{u}(t)$ on it's own previous values will benefit the leader.

$$\dot{\tilde{x}}(t) = \begin{bmatrix} A & 0 \\ \bar{A}_1 & \bar{A}_2 \end{bmatrix} \tilde{x}(t) + \begin{bmatrix} B_1 \\ \bar{B}_1 \end{bmatrix} \bar{u}(t) + \begin{bmatrix} B_2 \\ \bar{B}_2 \end{bmatrix} \bar{v}(t), \quad \tilde{x}(t_o) = \begin{bmatrix} x_o \\ z_o \end{bmatrix}$$ (4.94)

$$= \tilde{A}\tilde{x} + \tilde{B}_1\bar{u} + \tilde{B}_2\bar{v},$$

with costs J_1, J_2 as in (4.44), (4.45) and with the strategy of the leader restricted to be of the form

$$u(x_t, t) = \int_{t_o}^{t} [d_s \tilde{\eta}(t,s)] \tilde{x}(s) + L(t) \bar{v}(t).$$ (4.95)

The results of Section 4 are directly applicable to (4.94)-(4.95) and the problem is to find $\tilde{\eta}$, $\bar{A}_i$, $\bar{B}_i$, L, $\tilde{P}$ so that (4.63)-(4.65) are satisfied where in (4.63)-(4.65) one should use $\tilde{A}, \tilde{B}_1, (\tilde{B}_2 + \tilde{B}_1 L)$ in place of A, B_1, B_2. As far as it concerns z_o, it may be set arbitrarily equal to a constant or to a function of x_o preferably linear. The choice of z_o might affect not only the feasibility of (4.63)-(4.65) but the follower's optimum cost value as well. A simpler case of (4.95) is

$$\bar{u}(t) = \bar{L}_1 x(t) + \bar{L}_2 z(t) + L\bar{v}(t)$$ (4.96)

in which case the solution of the LF game is easy since the leader's controls are actually $\bar{A}_1$, $\bar{A}_2$, $\bar{B}_1$, $\bar{B}_2$, $\bar{L}_1$, $\bar{L}_2$, L, i.e. the leader plays open loop. Nonetheless the leader's problem will be nonlinear since his control multiplies the state $(x', z')'$.

4.6. A Nash Game with Memory

Consider the Nash game associated with (4.43)-(4.45) where at each instant of time t, both players have access to all the previous values of the state. The admissible strategies for both players are of the form

$$u(x_t, t) = \int_{t_o}^{t} [d_s \eta_1(t, s)] x(s) + b_1(t) \qquad (4.97)$$

$$v(x_t, t) = \int_{t_o}^{t} [d_s \eta_2(t, s)] x(s) + b_2(t). \qquad (4.98)$$

η_1 and η_2 are as in (4.35)-(4.36), $b_i(t)$ are piecewise continuous functions of time with appropriate dimensions. By x_t, $\bar{u}$, $\bar{v}$, we mean

$$\bar{u}(t) = u(x_t, t), \quad \bar{v}(t) = v(x_t, t) \qquad (4.99)$$

$$x_t: [to, t] \to R^n, \quad x_t(\theta), \quad \forall \theta \in [to, t], \quad \forall t \in [to, t_f]$$

In the next Proposition we give sufficient conditions for a pair of the form (4.97), (4.98) to constitute a Nash equilibrium pair. The first part of the Proposition refers to a particular initial point x_o, while the second part gives conditions similar to the coupled Riccati differential equations, see [1], which result in solutions in feedback form which are solutions for any initial point x_o.

<u>Proposition 4.1</u>: (i) Assume that there exist η_1^*, η_2^* as in (4.35)-(4.36), b_1^*, b_2^* piecewise continuous and μ_1, μ_2: $[t_o, t_f] \to R^n$ of bounded variation which satisfy:

$$\mu_i(t) - \int_t^{t_f} [A'(\tau)\mu_i(\tau) + Q_i(\tau)x(\tau)] d\tau + \int_t^{t_f} \eta_j^{*'}(\tau, t)[B_j(\tau)\mu_i(\tau)$$

$$+ R_{ij}(\tau)R_{jj}^{-1}(\tau)B_j'(\tau)\mu_j(\tau)] d\tau = F_i x(t_f), \quad i \neq j, \quad i, j = 1, 2 \qquad (4.100)$$

$$b_i^*(t) + \int_{t_o}^{t} [d_s \eta_i^*(t, s)] x(s) = -R_{ii}^{-1}(t)B_i'(t)\mu_i(t), i = 1, 2 \qquad (4.101)$$

$$\dot{x}(t) = A(t)x(t) - B_1(t)R_{11}^{-1}B_1'(t)\mu_1(t) - B_2(t)R_{22}^{-1}(t)B_2'(t)\mu_2(t) \qquad (4.102)$$

$$x(t_o) = x_o.$$

Then the strategies

$$u^*(x_t, t) = \int_{t_0}^{t} [d_s \eta_1^*(t,s)]x(s) + b_1^*(t) \tag{4.103}$$

$$v^*(x_t, t) = \int_{t_0}^{t} [d_s \eta_2^*(t,s)]x(s) + b_2^*(t) \tag{4.104}$$

constitute an equilibrium pair for the Nash game associated with (4.43)-(4.45) with admissible strategies (4.97), (4.98) and with $x(t_0) = x_0$.

(ii) Assume that there exist η_1^*, η_2^* as in (4.35)-(4.36) and matrix functions $P_1, P_2 : [t_0, t_f] \to R^n \times R^n$ of bounded variation which satisfy

$$P_i(t) - \int_{t}^{t_f} (A'(\tau)P_i(\tau) + Q_i(\tau) + \eta_j^{*'}(\tau,t)[B_j(\tau)P_i(\tau) + R_{ij}(\tau)R_{jj}^{-1}(\tau)B_j'(\tau)P_j(\tau)])\Phi(\tau,t)d\tau =$$

$$= F_i \Phi(t_f, t) \quad i,j = 1,2, \ i \neq j \tag{4.105}$$

$$\frac{\partial \Phi(t, t_0)}{\partial t} = [A(t) - B_1(t)R_{11}^{-1}(t)B_1'(t)P_1(t) - B_2(t)R_{22}^{-1}(t)B_2'(t)P_2(t)]\Phi(t, t_0) \tag{4.106}$$

$$\Phi(t_0, t_0) = I$$

$$\int_{t_0}^{t} [d_s \eta_i^*(t,s)] \Phi(s,t) = -R_{ii}^{-1}(t)B_2'(t)P_i(t), i = 1,2 . \tag{4.107}$$

Then the strategies

$$u^*(x_t, t) = \int_{t_0}^{t} [d_s \eta_1^*(t,s)]x(s) \tag{4.108}$$

$$v^*(x_t, t) = \int_{t_0}^{t} [d_s \eta_2^*(t,s)]x(s) \tag{4.109}$$

constitute an equilibrium pair for the Nash game associated with (4.43) - (4.45) with admissible strategies (4.97), (4.98) and for any $x_0 \in R^n$.

Proof: (i) If the second player plays (4.98) then (4.100) with i=1,j=2 and

$$\bar{u}(t) = -R_{11}^{-1}(t)B_2'(t)\mu_1(t) \qquad (4.110)$$

constitute sufficient conditions for optimality, by Theorem 4.1, of $\bar{u}$ for the control problem faced by the first player. (In (4.100) the term $R_{22}^{-1}B_2'\mu_2$ is substituted by $-\int_{t_o}^{t}[d_s\eta^*_2]x$ in these sufficient conditions.) Similar reasoning applies for the control problem faced by the second player when the first player plays (4.103).

(ii) We will first seek solutions μ_1,μ_2 of (4.100) which will work for any x_o: Let

$$\mu_i(t) = P_i(t)\Phi(t,t_o)x_o \qquad (4.111)$$

where Φ as in (4.106). Using (4.109) in (4.100) and (4.101) we obtain (4.105) and (4.107) where we considered $b_i \equiv 0$. It is clear now that if (4.105) and (4.107) hold, the μ_i's as in (4.107) satisfy (4.100)-(4.102). $\qquad \square$

The case where the players use strategies of the form

$$u(y_{1t},t) = \int_{t_o}^{t}[d_s\eta_1(t,s)]y_1(t,s) + b_1(t)$$

$$\qquad (4.112)$$

$$v(y_{2t},t) = \int_{t_o}^{t}[d_s\eta_2(t,s)]y_2(t,s) + b_2(t)$$

where for $i = 1,2$:

$$y_i(t,s) = C_i(t,s)x(s), \quad t_o \leq s \leq t,$$

$$\frac{d}{d_s}C_i(t,s) = 0 \quad a.e. \quad t_o \leq s \leq t \leq t_f,$$

$$y_i(t,s) \in R^{n_i}$$

and $\tilde{\eta}_i(t,s) = \eta_i(t,s)C_i(t,s)$ are as in (4.35)-(4.36) can also be considered.
The strategies (4.112) correspond to the case where the i-th player's information at time t is $\{C_i(t,s)x(s); \, t_o \leq s \leq t\}$. We only mention that in this case $\eta_i^* C_i$ should play the role of η_i^* in the conditions of Proposition 4.1

The results of Proposition 4.1 (see also problem (P')) can be used to study the Nash game associated with (4.43) - (4.45) where the players use previous values of their opponents strategy values. For example

$$\bar{u}(t) = \int_{t_o}^{t} [d_s \eta_{11}(t,s)] x(s) + \int_{t_o}^{t} [d_s \eta_{12}(t,s)] \bar{v}(s)$$

$$\bar{v}(t) = \int_{t_o}^{t} [d_s \eta_{21}(t,s)] x(s) + \int_{t_o}^{t} [d_s \eta_{22}(t,s)] \bar{u}(s).$$

Strategies of the form (4.92), (4.95), (4.96) can be considered for the Nash game and the augmentation (4.93)-(4.94) may also be employed in this case. The procedure for studying sufficiency conditions for Nash games with such strategies should be obvious by now and we will not take it up here.

CHAPTER 5

ON SOME STOCHASTIC STATIC AND DYNAMIC NASH GAMES

5.1. Introduction

In this chapter we deal with stochastic Nash games. We start in Section 5.2 by considering a static stochastic Nash game, where each player has a quadratic cost and his information is a linear function of a Gaussian random variable. Certain known results are considered first and some new ones are provided concerning the existence and uniqueness of the solution. In Section 5.3 we study solutions of the game of Section 5.2 which are affine in the information and provide a method for finding them. In Section 5.4 we generalize some of the results of the previous sections to the case where each player's control vector is subdivided into smaller control vectors, each one of which has to use different information. The information available to the subvectors of each control vector is nested. In Section 5.5 we consider a discrete time stochastic Nash game with linear stochastic state equation and quadratic costs where the players have noise corrupted state measurements. Special cases of this game were solved in [17] and [63]. The case where both players have perfect state measurements, i.e. $C_{i,k} = I$ and $v_{i,k} \equiv 0$ $\forall i,k$ (see (5.51) - (5.55)) was studied in [17] and it was shown that if the noise w_k in the state equation is nondegenerate, then the game admits a unique solution affine in the information under invertibility conditions for certain matrices. It was also shown in [17] that if w_k is degenerate, then the game will have in general an infinite number of nonlinear solutions. The case where at each stage k the players share their previous state measurements and their information

differ only in the k-th state measurement was studied in [63] where it was
shown that the game will admit a unique solution affine in the information
under invertibility conditions for certain matrices. In the more general
case where the assumptions of [17], [63] concerning the information of the
players do not hold the solutions of the game becomes extremely difficult.
In Section 5.5 we single out some new classes of problems which can be
relatively easily solved by using the results of the previous sections. In
the last Section 5.6 we translate some of the results of Section 5.5 to a
continuous time stochastic Nash game with a linear stochastic state equation
and quadratic costs where the players have noise corrupted state measurements.
Examples demonstrating certain properties of the solutions are also
given.

5.2. A Stochastic Static Nash Game

In this section we consider a static stochastic Nash game. After
a brief review of some results already available in the literature we
examine certain properties of this game and present some new results.

Let x be a Gaussian random variable over a probability space
$(\Omega, \mathcal{J}, P)$, $x : \Omega \to R^n$ with mean $\bar{x}$ and covariance Σ, and y_1, y_2 two random
variables defined by

$$y_i = H_i x, \qquad i = 1, 2 \tag{5.1}$$

where H_i is an $n_i \times n$ real constant matrix. Consider the Nash game with two
players, 1 and 2, where player i chooses a Borel measurable function
$u_i : R^{n_i} \to R^{m_i}$ and his cost is

$$J_i(u_1, u_2) = \mathop{E}_x[\frac{1}{2} u_i'(y_i)u_i(y_i) + u_i'(y_i)Q_ix + u_i'(y_i)R_iu_j(y_j) + u_i'(y_i)h_i]$$

$$i \neq j, \quad i,j = 1,2. \tag{5.2}$$

Q_i, R_i, h_i are real constant matrices of appropriate dimensions. y_i is referred to as the information available to player i. We also want u_1, u_2 to be chosen so that $u_1(y_1), u_2(y_2)$ have finite second moments. Thus our problem is to find (u_1^*, u_2^*) which satisfies

$$J_1(u_1^*, u_2^*) \leq J_1(u_1, u_2^*), \quad \forall \text{ admissible } u_1$$
$$J_2(u_1^*, u_2^*) \leq J_2(u_1^*, u_2), \quad \forall \text{ admissible } u_2. \tag{5.3}$$

A straightforward application of Radner's theorem [51],[52] results in the following theorem (Theorem 1 of [52]).

Theorem 5.1: (u_1^*, u_2^*) is a Nash equilibrium pair for the game defined above if and only if

$$u_1(y_1) = -\mathop{E}_x[Q_1x + R_1u_2(y_2) + h_1|y_1] \tag{5.4}$$
$$u_2(y_2) = -\mathop{E}_x[Q_2x + R_2u_1(y_1) + h_2|y_2]. \tag{5.5}$$

Substituting u_2 from (5.5) into (5.4) we obtain

$$u_1(y_1) = -Q_1E[x|y_1] - h_1 + R_1h_2 + R_1Q_2E[E[x|y_2]|y_1]$$
$$+ R_1R_2E[E[u_1(y_1)|y_2]|y_1]. \tag{5.6}$$

Therefore the investigation of solutions of this Nash game is equivalent to the investigation of solutions u_1 of (5.6). It can be easily shown that since the operator $P_i = E[\cdot|y_i]$ is a projection operator in an appropriately defined Banach space L ([52],[53]), it has norm ≤ 1, and thus the operator $P = R_1R_2E[E[\cdot|y_2]|y_1]$ has norm $\|P\| \leq \|R_1R_2\|$. Thus if $\|P\| \leq \|R_1R_2\| < 1$, the expansion $(I-P)^{-1} = I + P + P^2 + \cdots$ holds ([54], p. 231) and thus (5.6) has a unique solution given by

$$u_1(y_1) = (I + P + P^2 + \cdots)(-Q_1 P_1 x + R_1 Q_2 P_1 P_2 x - h_1 + R_1 h_2). \tag{5.7}$$

Since y_1 and y_2 are given by (5.1) and x is Gaussian we conclude that the converging infinite sum (5.7) is a sum of Gaussian random variables each one of which is affine in y_1 and thus we conclude that $u_1(y_1)$ will be of the form

$$u_1(y_1) = L_1 y_1 + d_1 \tag{5.8}$$

where L_1, d_1 are some constant real matrices. If (5.8) holds then we see from (5.5) that $u_2(y_2)$ will be of the same form, i.e.

$$u_2(y_2) = L_2 y_2 + d_2. \tag{5.9}$$

The above discussion is formalized in the following theorem of [52].

Theorem 5.2: If $\|R_1 R_2\| < 1$ or $\|R_2 R_1\| < 1$, then the Nash game has a unique solution which is of the form (5.8),(5.9).

Theorem 5.2 guarantees existence and uniqueness of a Nash solution and that the optimal strategies will be affine in the information. It is crucial to notice that the condition $\|R_1 R_2\| < 1$ has nothing to do with the H_1, H_2 matrices which nonetheless constitute a quite substantial part of the data of the problem. For example if H_1, H_2 are such that $P_1 P_2 = 0$, i.e. P_1, P_2 project into orthogonal subspaces of L then $P = 0$ and the solution u_1 of (5.6) is given by the first two terms of (5.6) and is of the form (5.8), whatever $R_1 R_2$ might be. More generally one can see that the solution of (5.6) is unique if and only if the operator $I - R_1 R_2 P_1 P_2$ is invertible.

Theorem 5.3: Assume that $P_1 P_2$ or $P_2 P_1$ is a projection. Then if $\det(I - R_1 R_2) \neq 0$, the Nash game admits a unique solution pair (u_1^*, u_2^*) with u_i^* affine in y_i.

<u>Proof</u>: Let $P_1P_2 = P$ be a projection. Then if $(I-R_1R_2P_1P_2)\varphi = 0 \Rightarrow (I-R_1R_2P)\varphi = 0$.

Let $\varphi = \varphi_1 + \varphi_2$, $P\varphi_1 = \varphi_1$, $P\varphi_2 = 0$. Thus $\varphi_1 + \varphi_2 = R_1R_2\varphi_1 \Rightarrow \varphi_2 = 0 \Rightarrow (I-R_1R_2)\varphi_1 = 0$.

Since $\det(I-R_1R_2) \neq 0 \Rightarrow \varphi_1 = 0$. Thus $\varphi = 0$ and $I-R_1R_2P_1P_2$ is invertible. By Theorems 1 and 2, p. 228 of ref. [54], we conclude that $(I-R_1R_2P_1P_2)^{-1}$ is a linear continuous operator. Thus u_1^* is affine in y_1 and consequently u_2^* is affine in y_2.

<u>Corollary 5.1</u>: If $H_2 = C_1H_1$ (or $H_1 = C_2H_2$) for some matrix C_1 (or C_2), then $P_1P_2 = P_2P_1 = P_2$, $(P_2P_1 = P_1P_2 = P_1)$.

<u>Proof</u>: Let $\mathfrak{F}_1, \mathfrak{F}_2$ be the minimal σ-fields in Ω with respect to which y_1, y_2 are respectively measurable. If $H_2 = C_1H_1 \Rightarrow \mathfrak{F}_2 \subseteq \mathfrak{F}_1 \Rightarrow E[E[\cdot|\mathfrak{F}_1]|\mathfrak{F}_2] = E[E[\cdot|\mathfrak{F}_2]|\mathfrak{F}_1] = E[\cdot|\mathfrak{F}_2]$ by Theorem 6.9, p. 260 of [55], i.e. $P_1P_2 = P_2P_1 = P_2$ and P_2 is a projection. ∎

Notice that if $\|R_1R_2\| < 1$ (or $\|R_2R_1\| < 1$), the expansion (5.7) holds while it does not have to hold under the assumptions of Theorem 5.3.

Let us now assume that for some nonzero vector e it holds $R_1R_2e = e$ and that H_1, H_2 are such that they provide a piece of common information to both players. Let us also assume without loss of generality, that x_1, the first component of x is available to both players. Consider any nonzero Borel measurable real valued function $\Psi(x)$. Then $P_1e \cdot \Psi(x_1) = P_2e \cdot \Psi(x_1) = e \cdot \Psi(x_1)$ and, $e \cdot \Psi(x_1) = R_1R_2e \cdot \Psi(x_1)$ and $e \cdot \Psi(x_1) \neq 0$. Thus, if (5.6) has a solution and $\det(I-R_1R_2) = 0$ this solution will not be unique. We thus conclude that the condition $\det(I-R_1R_2) = 0$ is necessary for the uniqueness of the solution of the Nash Game, if the two players share a common nontrivial part of their information.

We conclude this section with an example which demonstrates that the Nash game may have no solution at all or an infinite number of solutions.

Example 5.1: Let $x = (x_1, x_2)'$, y_1, y_2, u_1, u_2 take values in R^2, $\bar{x} = 0$, $h_1 = h_2 = 0$

$$cov(x) = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \tag{5.10}$$

and

$$y_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x, \qquad y_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x . \tag{5.11}$$

Then (5.6) assumes the form

$$(I - R_1 R_2) u_1 = (-Q_1 + R_1 Q_2) \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \tag{5.12}$$

Let

$$R_1 = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}, \qquad R_2 = \begin{bmatrix} 1 & 1/2 \\ 3 & 0 \end{bmatrix}$$

$$Q_1 = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} 1 & 2 \\ 1 & 7 \end{bmatrix}$$

where $1 = 2\beta - 5\alpha$. Then

$$I - R_1 R_2 = \begin{bmatrix} -2 & 0 \\ -5 & 0 \end{bmatrix}$$

$$-Q_1 + R_1 Q_2 = \begin{bmatrix} 1-\alpha & 7-\gamma \\ 3-\beta & 11-\delta \end{bmatrix}$$

and (5.12) results in

$$u_1^*(x_1) = \begin{bmatrix} \dfrac{\alpha-1}{2} x_1 \\ \varphi(x_1) \end{bmatrix} \tag{5.13}$$

where $\varphi(x_1)$ is any Borel measurable function of x_1 which results in $u_1(x_1)$ having finite covariance. (If we had $1 \neq 2\beta - 5\alpha$ then (5.12) would have no solution.) From (5.5) we obtain

$$u_2^*(x) = - \begin{bmatrix} \frac{\alpha+1}{2} + \frac{1}{2} \varphi(x_1) + 2x_2 \\[2ex] \frac{3\alpha-1}{2} x_1 + 7x_2 \end{bmatrix} \tag{5.14}$$

and using (5.2), (5.13), and (5.14) we find

$$J_1^* = -\frac{1}{2}(\alpha-1)^2\sigma_1 - \frac{1}{2}\mathop{E}_{x_1}[\varphi^2(x_1)] \tag{5.15}$$

$$J_2^* = -\frac{6\alpha^2-2\alpha+1}{4}\sigma_1 - \frac{1}{8}\mathop{E}_{x_1}[\varphi^2(x_1) + 2x_1\varphi(x_1)]. \tag{5.16}$$

Notice that by choosing $\varphi_n(x_1) = nx_1$ and letting $n \to +\infty \Rightarrow J_1^* \to -\infty$. Notice also that the solution of the game is exactly the same even if in the cost functional of player 1, terms of the form $u_2'u_2$, $u_2'x$, $x'x$ were included. Actually by adding to the cost functional of player 1 the terms $\frac{1}{2}\theta u_2'u_2$, $\frac{1}{2}\theta x'x$, where $\theta > 0$ we see that J_1 becomes a quadratic of the form

$$\frac{1}{2}[u_1',u_2',x'] \begin{bmatrix} I & R_1 & Q_1 \\ R_1' & \theta I & 0 \\ Q_1' & 0 & \theta I \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ x \end{bmatrix}$$

which is positive semidefinite for a sufficiently large θ (see [56]). Addition of such terms to J_1 will not alter the solution, but will alter J_1^*, J_2^* and thus will eliminate the possibility $J_1^* \to -\infty$. Finally notice that $I-R_1R_2$ is singular and $H_1 = C_2H_2$ and thus the possibility of nonunique nonlinear strategies was expected by the discussion following Corollary 5.1.

5.3. Affine Solutions of the Stochastic Static Nash Game

In the previous section we stated conditions under which the Nash game posed there will have a unique solution with the controls of the players being affine in their information. The conditions $\det(I-R_1R_2) \neq 0$

and $H_2 = C_1 H_1$ demand some special attention. Let us assume that $H_2 = C_1 H_1$.

If it happens that $\det(I-R_1 R_2) = 0$, then by perturbing very little the

arguments of R_1 and R_2, we can guarantee that for the perturbed pair

$(\overline{R}_1, \overline{R}_2)$, $\det(I-\overline{R}_1 \overline{R}_2) \neq 0$ will hold. This is important because in a real life

problem the values of R_1, R_2 are approximately known. Thus almost all

such Nash games admit a unique solution which is affine in the information.

On the other hand if $\det(I-R_1 R_2) = 0$ but it is very close to zero, it might

happen that out of the possibly infinite solutions one can result in making

J_1 or J_2 arbitrarily small, (recall Example 5.1), i.e. if $\det(I-R_1 R_2) \cong 0$

the game might exhibit an unstable behavior. Therefore, in order to

guarantee uniqueness of an affine solution in a realistic framework, one

should impose a condition of the form $|\det(I-R_1 R_2)| \geq \varepsilon > 0$, where ε is some

positive constant which depends on the measurement error of the entries of

R_1 and R_2. The condition $H_2 = C_1 H_1$ can be satisfied if a negligible but

nonzero signaling is allowed between the players (see discussions in [57],

[58], [59]) and this in conjunction with $\det(I-R_1 R_2) \neq 0$ will guarantee

existence of a unique solution affine in the information.

Let us now restrict a priori our attention to admissible pairs

(u_1, u_2) of the form (5.8),(5.9). It is clear that if player 1 plays (5.8)

then player 2 will use a strategy of the form (5.9) and vice versa (see [51]).

In this case we have the following theorem.

Theorem 5.4: Assume that H_1, H_2 have full rank n_1 and n_2 respectively, and

$\Sigma > 0$, so that $H_1 \Sigma H_1'$, $H_2 \Sigma H_2'$ are invertible. Then, a pair (u_1, u_2) of the form

$(L_1 y_1 + d_1, L_2 y_2 + d_2)$ is a solution of the Nash game posed in Section 5.2,

if and only if the following matrix equations are satisfied.

$$L_1 = -(Q_1 + R_1 L_2 H_2) \Sigma H_1' (H_1 \Sigma H_1')^{-1} \tag{5.17}$$

$$L_2 = -(Q_2 + R_2 L_1 H_1) \Sigma H_2' (H_2 \Sigma H_2')^{-1} \tag{5.18}$$

$$d_1 = -(Q_1 + R_1 L_2 H_2)(I - \Sigma H_1'(H_1 \Sigma H_1')^{-1} H_1)\bar{x} - (R_1 d_2 + h_1) \tag{5.19}$$

$$d_2 = -(Q_2 + R_2 L_1 H_1)(I - \Sigma H_2'(H_2 \Sigma H_2')^{-1} H_2)\bar{x} - (R_2 d_1 + h_2). \tag{5.20}$$

$\underline{\text{Proof}}$: The proof is a matter of substitution of $u_i = L_i y_i + d_i$ in (5.4),(5.5) and identification of the appropriate matrices. ∎

Substitution of L_2 from (5.18) in (5.17) results in

$$L_1 - R_1 R_2 L_1 H_1 \Sigma H_2'(H_2 \Sigma H_2')^{-1} H_2 \Sigma H_1'(H_1 \Sigma H_1')^{-1} = -Q_1 \Sigma H_1'(H_1 \Sigma H_1')^{-1}$$
$$+ R_1 Q_2 \Sigma H_2'(H_2 \Sigma H_2')^{-1} H_2 \Sigma H_1'(H_1 \Sigma H_1')^{-1}. \tag{5.21}$$

Since H_1 has full rank, by introducing $M_1 = L_1 H_1$ we can write (5.21) equivalently as

$$M_1 - R_1 R_2 M_1 \Sigma H_2'(H_2 \Sigma H_2')^{-1} H_2 \Sigma H_1'(H_1 \Sigma H_1')^{-1} H_1 = -Q_1 \Sigma H_1'(H_1 \Sigma H_1')^{-1} H_1$$
$$+ R_1 Q_2 \Sigma H_2'(H_2 \Sigma H_2')^{-1} H_2 \Sigma H_1'(H_1 \Sigma H_1')^{-1} H_1 \tag{5.22}$$

Notice that $u_1 = L_1 y_1 + d_1 = M_1 x + d_1$. Substitution of d_2 from (5.20) in (5.19) results in

$$d_1 - R_1 R_2 d_1 = -(Q_1 + R_1 L_2 H_2)(I - \Sigma H_1'(H_1 \Sigma H_1')^{-1} H_1)\bar{x} - h_1$$
$$+ R_1(Q_2 + R_2 L_1 H_1)(I - \Sigma H_2'(H_2 \Sigma H_2')^{-1} H_2)\bar{x} + R_1 h_2. \tag{5.23}$$

Notice that as soon as M_1 is found from (5.22), we obtain $L_1 = M_1(H_1 H_1')^{-1}$. L_2 is determined from (5.18) and with L_1, L_2 given, the solution of (5.23) for d_1 determines d_2 from (5.20). Consequently the study of solutions affine in the information is reduced to the study of solutions of (5.22) and (5.23). Obviously, if $\det(I - R_1 R_2) \neq 0$ (5.23) results in a unique solution for d_1 in terms of M_1. Therefore we will focus on (5.22).

Lemma 5.1: If for any pair (λ,μ) of eigenvalues of $R_1 R_2$ and

$\Sigma H_2'(H_2 \Sigma H_2')^{-1} H_2 \Sigma H_1'(H_1 \Sigma H_1')^{-1} H_1$ respectively, $1 \neq \lambda\mu$, then (5.22) has a unique

solution in M_1.

Proof: The proof is an application of Theorem 1 of [60]. ∎

Notice that interchange of the indices 1 and 2 in the hypothesis of Lemma 5.1

does not result in any benefit since for any matrices A,B, which commute

AB and BA have the same eigenvalues.

Proposition 5.1: If $\det(I-R_1 R_2) \neq 0$ and $H_1 = C_2 H_2$ or $H_2 = C_1 H_1$, then (5.17)-

(5.20) has a unique solution in L_1, L_2, d_1, d_2.

Proof: If $H_1 = C_2 H_2$ the product $\Sigma H_2'(H_2 \Sigma H_2')^{-1} H_2 \Sigma H_1'(H_1 \Sigma H_1')^{-1} H_1 =$

$= \Sigma H_2'(H_2 \Sigma H_2')^{-1} H_2 \Sigma H_2' C_2'(H_1 \Sigma H_1')^{-1} H_1 = \Sigma H_2' C_2'(H_1 \Sigma H_1')^{-1} H_1 = \Sigma H_1'(H_1 \Sigma H_1')^{-1} H_1$,

i.e. it is a projection and thus it has eigenvalues 0 or 1. Since

$\det(I-R_1 R_2) \neq 0$ we conclude that all eigenvalues of $R_1 R_2$ are different than 1

and thus the hypothesis of Lemma 5.1 holds. Thus L_1, L_2 can be found

uniquely. Since $\det(I-R_1 R_2) \neq 0$ (5.23) can be solved uniquely for d_1. ∎

Clearly the conclusion of Proposition 5.1 was expected in view of

Theorem 5.3 and Corollary 5.1. The above results provide conditions for exis-

tence of a unique solution affine in the information. If these conditions do

not hold, we might have no affine solutions at all or an infinite number of

affine solutions. Notice that in our results the matrices H_1, H_2 play a

certain role while in Theorem 5.2 they do not.

From equation (5.22) it is obvious that the study of solutions

affine in the information is intimately related with the study of the matrix

equation

$$Y - AYB = D. \tag{5.24}$$

In the sequel we will deal in more detail with this equation. Let $\Lambda_a = \text{diag}(A_1,\ldots,A_p)$, $\Lambda_b = \text{diag}(B_1,\ldots,B_q)$ be the Jordan forms of A and B respectively, with $A = T\Lambda_a T^{-1}$, $B = R\Lambda_b R^{-1}$. We set $X = T^{-1}YR$, $C = T^{-1}DR$ and thus (5.24) is equivalent to

$$X - \Lambda_a X \Lambda_b = C. \tag{5.25}$$

We partition X and C in the form $X = (X_{ij})$, $C = (C_{ij})$, $i = 1,\ldots,p$, $j = 1,\ldots,q$, so that (5.25) is equivalent to the system (5.26) of the $p \cdot q$ independent matrix equations,

$$X_{ij} - A_i X_{ij} B_j = C_{ij}, \quad i = 1,\ldots,p, \ j = 1,\ldots,q. \tag{5.26}$$

We have

$$A_i = \lambda_i I + H_i, \quad B_j = \mu_j I + E_j$$

where H_i, E_j are of the form

$$\begin{bmatrix} 0 & 1 & 0 & \ldots\ldots & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 \\ \vdots & & \ddots & \ddots & & \\ \vdots & & & \ddots & \ddots & \\ 0 & \ldots\ldots\ldots & & 0 & 1 \\ 0 & \ldots\ldots\ldots\ldots & & & 0 \end{bmatrix}$$

with dimensions $m_i \times m_i$, $n_j \times n_j$ respectively. It holds $H_i^{m_i} = 0$, $E_j^{n_j} = 0$.

Case 1: $1 = \lambda_i \mu_j$. (5.26) can be written (we drop the indices for convenience)

$$X = \frac{1}{1-\lambda\mu} [C + \lambda XE + \mu HX + HXE]. \tag{5.27}$$

Multiplying (5.27) by H,E we obtain

$$XE = \frac{1}{1-\lambda\mu} [CE + \lambda XE^2 + \mu HXE + HXE^2] \tag{5.28}$$

$$HX = \frac{1}{1-\lambda\mu} [HC + \lambda HXE + \mu H^2X + H^2XE] \tag{5.29}$$

$$HXE = \frac{1}{1-\lambda\mu} [HCE + \lambda HXE^2 + \mu H^2XE + H^2XE^2] \tag{5.30}$$

We substitute HXE from (5.30) into (5.28) and (5.29) and then substitute
XE, HX, HXE from (5.28)-(5.30) into (5.27). We obtain

$$X = \alpha_1 C + \alpha_2 CE + \alpha_3 HC + \alpha_4 HCE + \alpha_5 XE^2 + \alpha_6 HXE^2 + \alpha_7 H^2 X$$
$$+ \alpha_8 H^2 XE + \alpha_9 H^2 XE^2 \tag{5.31}$$

where $\alpha_1, \ldots, \alpha_9$ are functions of λ_i and μ_j. The importance of (5.31) is
that in the terms in the right hand side X is multiplied by E^2 or H^2.
Repeating the procedure but starting with (5.31) instead of (5.27), after ρ
steps we obtain an expression of X in terms of $H^k CE^\ell$ and $H^\sigma XE^\tau$, $\sigma, \tau \geqslant \rho$.
If $\rho \geqslant \min(m_i, n_j)$ then the terms $H^\sigma XE^\tau$ are zero and thus after e steps we
have obtained the unique solution of (5.26). The existence of a unique
solution could be foreseen by Theorem 1 of [60], since $1 \neq \lambda_i \mu_j$.

<u>Case 2</u>: $1 = \lambda_i \mu_j$. In this case A_i is nonsingular and (5.26) can be written as

$$A_i^{-1} X_{ij} - X_{ij} B_j = A_i^{-1} C_{ij}. \tag{5.32}$$

(5.32) has a solution if and only if the matrices

$$\begin{pmatrix} A_i^{-1} & A_i^{-1} C_{ij} \\ 0 & B_j \end{pmatrix} \quad , \quad \begin{pmatrix} A_i^{-1} & 0 \\ 0 & B_j \end{pmatrix}$$

are similar, [61]. If similarity holds then (5.32) has at least one
solution. Since the homogeneous equation

$$X_{ij} - A_i X_{ij} B_j = 0 \tag{5.33}$$

has $1 = \lambda_i \mu_j$, it will have an infinite number of solutions and so will (5.32).
Methods for solution of (5.32) are known and we will not take up this issue
here.

5.4. Generalizations

In this section we will prove generalizations of Theorems 5.2 and 5.3. Consider the Nash game where player 1 chooses $(u_1,\ldots,u_N)$, player 2 chooses $(v_1,\ldots,v_N)$, their respective costs are

$$J_1 = \underset{x}{E}[\frac{1}{2}\sum_{i=1}^{N}u_i'u_i + \sum_{i=1}^{N}u_i'Q_ix + \sum_{i,j=1}^{N}u_i'R_{ij}v_j + \frac{1}{2}\sum_{\substack{i,j=1\\i\neq j}}^{N}u_i'\Gamma_{ij}u_j] \tag{5.34}$$

$$J_2 = \underset{x}{E}[\frac{1}{2}\sum_{i=1}^{N}v_i'v_i + \sum_{i=1}^{N}v_i'\overline{Q}_ix + \sum_{i,j=1}^{N}v_i'\overline{R}_{ij}u_j + \frac{1}{2}\sum_{\substack{i,j=1\\i\neq j}}^{N}v_i'\overline{\Gamma}_{ij}v_j] \tag{5.35}$$

where x is a Gaussian random variable and x,u_i,v_i takes values in finite dimensional Euclidean spaces. u_i is to be chosen as a Borel measurable function of

$$Z_i = \{y_1,\ldots,y_i\}, \quad y_i = H_ix + \sum_{j<i}L_{ij}u_j(Z_j) \tag{5.36}$$

and v_i is to be chosen as a Borel measurable function of

$$\overline{Z}_i = \{\overline{y}_1,\ldots,\overline{y}_i\}, \quad \overline{y}_i = \overline{H}_ix + \sum_{j<i}\overline{L}_{ij}v_j(\overline{Z}_j). \tag{5.37}$$

In addition, u_i,v_i must have finite second moments. Notice that since the information structure of each player is nested, see [62], we can actually disregard the $\sum_{j<i}L_{ij}u_j$, $\sum_{j<i}\overline{L}_{ij}v_j$ terms in (5.36),(5.37). The matrices $Q_i,\overline{Q}_i,R_{ij},\overline{R}_{ij}$, $\Gamma_{ij}=\Gamma_{ji}'$, $\overline{\Gamma}_{ij}=\overline{\Gamma}_{ji}'$, $H_i,\overline{H}_i$, $L_{ij},\overline{L}_{ij}$, are real, constant, of appropriate dimensions.

__Theorem 5.5:__ A pair $((u_1^*,\ldots,u_N^*),(v_1^*,\ldots,v_N^*))$ constitutes a solution of the Nash game associated with (5.34)-(5.37) if

$$u_i^* = -E[Q_ix + \sum_j R_{ij}v_j + \sum_{j\neq i}\Gamma_{ij}u_j|Z_i] \tag{5.38}$$

$$v_i^* = -E[\overline{Q}_ix + \sum_j \overline{R}_{ij}u_j + \sum_{j\neq i}\overline{\Gamma}_{ij}v_j|\overline{Z}_i]. \tag{5.39}$$

__Proof:__ The proof is in spirit the same to the one of Theorem 1 in [52]. ∎

If we set $E[\cdot|Z_i] = P_i$, $E[\cdot|\bar{Z}_i] = \bar{P}_i$ and take into account that $E[u_i|Z_j] = u_i$, $E[v_i|\bar{Z}_j] = v_i$, if $i \leq j$, we conclude that (5.38) and (5.39) can be written as

$$
\left[
\begin{array}{ccccc|cccc}
I & \Gamma_{12}P_1 & \Gamma_{13}P_1 & \cdots & \Gamma_{1N}P_1 & R_{11}P_1 & R_{12}P_1 & \cdots & R_{1N}P_1 \\
\Gamma_{21} & I & \Gamma_{23}P_2 & \cdots & \Gamma_{2N}P_2 & R_{21}P_2 & R_{22}P_2 & \cdots & R_{2N}P_2 \\
\vdots & & & & \vdots & \vdots & & & \vdots \\
\Gamma_{N1} & \Gamma_{N2} & \cdots\, \Gamma_{N,N-1} & & I & R_{N1}P_N & \cdots\cdots\cdots & & R_{NN}P_N \\
\hline
\bar{R}_{11}\bar{P}_1 & \cdots\cdots\cdots\cdots & & & \bar{R}_{1N}\bar{P}_1 & I & \bar{\Gamma}_{12}\bar{P}_1 & \cdots\cdots & \bar{\Gamma}_{1N}\bar{P}_1 \\
\vdots & & & & \vdots & \bar{\Gamma}_{21} & I & \bar{\Gamma}_{23}\bar{P}_2\cdots\bar{\Gamma}_{2N}\bar{P}_2 \\
\vdots & & & & \vdots & \vdots & & & \vdots \\
\bar{R}_{N1}\bar{P}_N & \cdots\cdots\cdots\cdots & & & \bar{R}_{NN}\bar{P}_N & \bar{\Gamma}_{N1} & \bar{\Gamma}_{N2} & \cdots & I
\end{array}
\right]
\left[
\begin{array}{c}
u_1 \\ u_2 \\ \vdots \\ u_N \\ v_1 \\ \vdots \\ \vdots \\ v_N
\end{array}
\right]
=
$$

$$
= -\left[
\begin{array}{c}
Q_1 P_1 \\
Q_2 P_2 \\
\vdots \\
Q_N P_N \\
\bar{Q}_1 \bar{P}_1 \\
\vdots \\
\bar{Q}_N \bar{P}_N
\end{array}
\right] x
\tag{5.40}
$$

or in

$$
(I - \tilde{L})u = \tilde{T}x. \tag{5.41}
$$

Since all the projections $P_i, \bar{P}_i$ have norm ≤ 1, one can readily conclude that if the matrices $\Gamma_{ij}, \bar{\Gamma}_{ij}, R_{ij}, \bar{R}_{ij}$ are chosen sufficiently small, then the operator $\tilde{L}$ has norm < 1 and thus $(I - \tilde{L})^{-1} = I + \tilde{L} + \tilde{L}^2 + \cdots$. Thus u can be expressed as a convergent infinite sum of Gaussian random variables affine in x and thus u is affine in x. We thus have proved the following theorem.

Theorem 5.6: If the norms of $\Gamma_{ij}, \bar{\Gamma}_{ij}, R_{ij}, \bar{R}_{ij}$ are sufficiently small, then the solution of the Nash game (5.34)-(5.37) exists, is unique, and each u_i, v_i is affine in its information.

In the last part of this section we will show how Theorem 5.3 and Corollary 5.1 can be generalized for the problem (5.34)-(5.37). In order to illustrate the procedure, let us assume that $N = 2$. Let $P_1 = E[\cdot | Z_1]$, $P_2 = E[\cdot | Z_2]$, $\bar{P}_1 = E[\cdot | \bar{Z}_1]$, $\bar{P}_2 = E[\cdot | \bar{Z}_2]$. Let us also assume that for some matrices C_1, C_2 it holds $H_1 = C_1 \bar{H}_1$, $H_2 = C_2 \bar{H}_1$ (recall assumptions of Corollary 5.1). Also let $L_{ij} = 0$, $\bar{L}_{ij} = 0$ without loss of generality. Then $\bar{P}_1 P_1 = P_1 \bar{P}_1 = P_1$, $\bar{P}_1 P_2 = P_2 \bar{P}_1 = P_2$, $\bar{P}_2 P_2 = P_2 \bar{P}_2 = P_2$ (see Theorem 6.5.10, p. 260 in [55]). (5.38) and (5.39) assume the forms

$$u_1 = -Q_1 P_1 x - \Gamma_{12} P_1 u_2 - R_{11} P_1 v_1 - R_{12} P_1 v_2 \tag{5.42}$$

$$u_2 = -Q_2 P_2 x - \Gamma_{21} u_1 - R_{21} P_2 v_1 - R_{22} P_2 v_2 \tag{5.43}$$

$$v_1 = -\bar{Q}_1 \bar{P}_1 x - \bar{\Gamma}_{12} \bar{P}_1 v_2 - \bar{R}_{11} \bar{P}_1 u_1 - \bar{R}_{12} \bar{P}_1 u_2 \tag{5.44}$$

$$v_2 = -\bar{Q}_2 \bar{P}_2 x - \bar{\Gamma}_{21} v_1 - \bar{R}_{21} \bar{P}_2 u_1 - \bar{R}_{22} \bar{P}_2 u_2. \tag{5.45}$$

Since $P_1 u_1 = u_1$, $P_2 u_2 = u_2 \Rightarrow \bar{P}_1 u_1 = \bar{P}_1 P_1 u_1 = P_1 u_1 = u_1$, $\bar{P}_1 u_2 = \bar{P}_1 P_2 u_2$, $\bar{P}_2 u_1 = \bar{P}_2 P_1 u_1 = P_1 u_1 = u_1$, $\bar{P}_2 u_2 = \bar{P}_2 P_2 u_2 = P_2 u_2 = u_2$, and since $\bar{\Gamma}_1 \bar{P}_2 = \bar{P}_1$, (5.44) and (5.45) become

$$v_1 = (I - \bar{\Gamma}_{12} \bar{\Gamma}_{21})^{-1} (-\bar{Q}_1 - \bar{\Gamma}_{12} \bar{Q}_2) \bar{P}_1 x + (I - \bar{\Gamma}_{12} \bar{\Gamma}_{21})^{-1} (-\bar{R}_{11} + \bar{\Gamma}_{12} \bar{R}_{21}) u_1$$
$$+ (I - \bar{\Gamma}_{12} \bar{\Gamma}_{21})^{-1} (-\bar{R}_{12} + \bar{\Gamma}_{12} \bar{R}_{22}) u_2 \tag{5.46}$$

$$u_2 = -\bar{Q}_2 \bar{P}_2 x - \bar{R}_{21} u_1 - \bar{R}_{22} u_2 - \bar{\Gamma}_{21} (I - \bar{\Gamma}_{12} \bar{\Gamma}_{21})^{-1} (-\bar{Q}_1 - \bar{\Gamma}_{12} \bar{Q}_2) \bar{P}_1 x$$
$$- \bar{\Gamma}_{21} (I - \bar{\Gamma}_{12} \bar{\Gamma}_{21})^{-1} (-\bar{R}_{11} + \bar{\Gamma}_{12} \bar{R}_{21}) u_1$$
$$- \bar{\Gamma}_{21} (I - \bar{\Gamma}_{12} \bar{\Gamma}_{21})^{-1} (-\bar{R}_{12} + \bar{\Gamma}_{12} \bar{R}_{22}) u_2 \tag{5.47}$$

where we assume that $\det(I-\bar{\Gamma}_{12}\bar{\Gamma}_{21}) \neq 0$. Substituting v_1, v_2 from (5.46), (5.47) into (5.42),(5.43) we obtain a system of the form

$$u_1 = A_1 P_1 x + A_2 P_1 u_2 + A_3 u_1 \tag{5.48}$$

$$u_2 = B_1 P_2 x + B_2 u_2 + B_3 u_1 \tag{5.49}$$

where A_i, B_i are matrices calculated after the substitution. If $\det(I-B_2) \neq 0$, we can solve (5.49) for u_2, substitute in (5.48) and obtain

$$u_1 = AP_1 x + BP_1 u_1 \tag{5.50}$$

where A, B are again matrices calculated after the substitution. If $\det(I-B) \neq 0$, then the operator $I-BP_1$ is invertible and (5.50) has a unique solution in u_1, linear in y_1. Therefore u_2, v_1, v_2 can be found uniquely by substituting u_1 in (5.49) and u_1, u_2 in (5.46),(5.47). We have thus shown that if certain matrices are invertible, then the game will have a unique solution, linear in the information under the crucial hypothesis that Z_2 is a linear transformation of $\bar{Z}_1$. The generalization of this procedure to the case here $N \geqslant 2$ is obvious. We can thus state the following theorem.

<u>Theorem 5.7</u>: If Z_N is a linear transformation of $\bar{Z}_1$, or $\bar{Z}_N$ is a linear transformation of Z_1, then if certain matrices are invertible, the game (5.34)-(5.37) has a unique solution linear in the information.

5.5. A Discrete-Time Stochastic Linear Quadratic Nash Game

In this section we consider a linear quadratic discrete time, stochastic, Nash game, where the players have noise corrupted measurements of the state. The aim of this section is to point out several difficulties and provide solutions or conditions for existence of solutions for some special cases.

Consider the dynamic system whose state x_k evolves in accordance with the equation

$$x_{k+1} = A_k x_k + B_{1,k} u_{1,k} + B_{2,k} u_{2,k} + w_k, \quad k = 0, 1, \ldots, N \quad (5.51)$$

where $u_{1,k}$, $u_{2,k}$ are chosen by two players who play Nash and whose respective costs are J_1, J_2,

$$J_1 = E[x'_{N+1} Q_{1,N+1} x_{N+1} + \sum_{k=0}^{N} (x'_k Q_{1,k} x_k + u'_{1,k} u_{1k} + u'_{2,k} R_{1,k} u_{2,k})] \quad (5.52)$$

$$J_2 = E[x'_{N+1} Q_{2,N+1} x_{N+1} + \sum_{k=0}^{N} (x'_k Q_{2,k} x_k + u'_{2,k} u_{2,k} + u'_{1,k} R_{2,k} u_{1,k})] \quad (5.53)$$

where $Q_{1,k} = Q'_{i,k} \geqslant 0$, $R_{i,k} = R'_{i,k} \geqslant 0$. $u_{i,k}$ is to be chosen as a Borel measurable function of z_k^i, where

$$z_k^i = \{y_{i,0}, y_{i,1}, \ldots, y_{i,k}\} \quad (5.54)$$

$$y_{i,k} = C_{i,k} x_k + v_{i,k}, \quad i = 1, 2. \quad (5.55)$$

$x_k, u_{1,k}, u_{2,k}$ take values in R^n, R^{m_1}, R^{m_2} respectively, w_k, $v_{i,k}$, and x_0 are independent Gaussian random vectors, the expectations in (5.43),(5.44) are with respect to all random variables and all the matrices have appropriate dimensions.

Some special cases of this game have been treated successfully, [17], [63], but in its general form, is quite difficult to solve. Only one class of admissible Nash strategies can be relatively easily investigated and this is the class of strategies affine in the information. If player 1 uses an affine function of his information z_k^1 at each stage k, i.e. $u_{1,k} = M_{1,k} z_k^1 + d_{1,k}$ then player 2 is faced with a classical LQG problem which he can easily solve and his optimal reaction can be restricted to be of the form $u_{2,k} = M_{1,k} z_k^2 + d_{2,k}$ without any deterioration to his cost J_2. The converse also holds. Thus we end up with

a system of matrix equations with unknowns $M_{i,k}, d_{i,k}$, $i=1,2$, $k=0,\ldots,N$. This system might have one, many, or no solution at all. The following example demonstrates that there might exist no solution in the affine class of strategies.

<u>Example 5.2</u>: Consider the one stage game

$$x_1 = Ax_o + B_1 u_1 + B_2 u_2$$

$$J_1 = E[x_1' Q_1 x_1 + u_1' u_1 + u_2' R_1 u_2]$$

$$J_2 = E[x_1' Q_2 x_1 + u_2' u_2 + u_1' R_2 u_1]$$

where $x_0 = (x_1, x_2)'$ has zero mean and covariance equal to the unit matrix, and x, u_1, u_2 take values in R^2. u_1 and u_2 have to be chosen as functions of $y_1 = C_1 x_0$ and $y_2 = C_2 x_0$ respectively. Substituting x_1 with its equal in J_1, J_2 we obtain a static problem with

$$\tilde{J}_1 = E[u_1' \frac{I + B_1' Q_1 B_1}{2} u_1 + u_1' B_1' Q_1 A x_0 + u_1' B_1' Q_1 B_2 u_2] \tag{5.56}$$

$$\tilde{J}_2 = E[u_2' \frac{I + B_2' Q_2 B_2}{2} u_2 + u_2' B_2' Q_2 A x_0 + u_2' B_2' Q_2 B_1 u_1]. \tag{5.57}$$

The solution (u_1, u_2) is given by

$$u_1 = -(I + B_1' Q_1 B_1)^{-1} B_1' Q_1 A E[x_0 | y_1] - (I + B_1' Q_1 B_1)^{-1} B_1' Q_1 B_2 E[u_2 | y_1] \tag{5.58}$$

$$u_2 = -(I + B_2' Q_2 B_2)^{-1} B_2' Q_2 A E[x_0 | y_2] - (I + B_2' Q_2 B_2)^{-1} B_2' Q_2 B_1 E[u_1 | y_2]. \tag{5.59}$$

Substitution of u_2 from (5.59) into (5.58) we obtain

$$\begin{aligned} u_1 = &-(I + B_1' Q_1 B_1)^{-1} B_1' Q_1 A E[x_0 | y_1] \\ &+ (I + B_1' Q_1 B_1)^{-1} B_1' Q_1 B_2 (I + B_2' Q_2 B_2)^{-1} B_2' Q_2 A E[E[x_0 | y_2] | y_1] \\ &+ (I + B_1' Q_1 B_1)^{-1} B_1' Q_1 B_2 (I + B_2' Q_2 B_2)^{-1} B_2' Q_2 B_1 E[E[u_1 | y_2] | y_1]. \end{aligned} \tag{5.60}$$

Let us now choose

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} \alpha & 0 \\ \beta & 1 \end{bmatrix}, \qquad C_1 = C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Then (5.60) assumes the form

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} u_1 = \begin{bmatrix} \frac{\beta - \alpha}{2} x_1 \\ 0 \end{bmatrix}$$

which is impossible to solve for u_1 if $\beta \neq \alpha$.

Since the information of each player is affected by the control laws of the other, one would be tempted to consider a subclass of problems where this is not the case. Let us assume that

$$C_{i,1} B_{j,0} = 0 \; \cdot$$

$$C_{i,2} [B_{j,1} \vdots A_1 B_{j,0}] = 0$$

$$C_{i,3} [B_{j,2} \vdots A_2 B_{j,1} \vdots A_2 A_1 B_{j,0}] = 0$$

$$\vdots$$

$$C_{i,k} [B_{j,k-1} \vdots A_{k-1} B_{j,k-2} \vdots \dots \vdots A_{k-1} A_{k-2} \cdots A_1 B_{j,0}] = 0 \qquad (5.61)$$

$$\vdots$$

$$C_{1,N} [B_{j,N-1} \vdots A_{N-1} B_{j,N-2} \vdots \dots \vdots A_{N-1} \cdots A_1 B_{j,0}] = 0$$

$$i \neq j, \qquad i,j = 1,2.$$

Equation (5.61) means that the controllability subspace of player j is orthogonal to the observability subspace of the $C_{i,k}$ matrices of player i. If (5.61) holds, then substitution of x_{k+1} in terms of $x_0, u_{1,0}, \dots, u_{1,k}$, $u_{2,0}, \dots, u_{2,k}$, for $k=0,\dots,N$ (by using recursively (5.51)), in the cost functionals J_1, J_2 will result in a Nash game of the type (5.34)–(5.37). The L_{ij} matrices in (5.30) are given by products and sums of the $C_{1,p}$, $A_p, B_{1,p}$ $(0 \leq p < j)$ matrices and the H_i matrices by products of the

$C_{1,p}, A_p$, $0 \leqslant p < j$ matrices. Similarly for the $\bar{L}_{ij}, \bar{H}_i$ matrices. Of course

the $v_{i,k}$, w_k noises can be incorporated together with x_0 in a random vector

which will play the role of x in (5.34)-(5.37). Therefore Theorem 5.6 can

be used to derive conditions which guarantee existence and uniqueness of a

Nash pair, affine in the information for the game (5.51)-(5.55) under the

hypothesis that (5.61) holds. Nonetheless, since for the Example 5.2

the condition (5.61) holds trivially, we conclude that (5.61) alone does not

in general suffice to guarantee existence of a Nash equilibrium pair.

Although (5.61) is a quite restrictive assumption (since for

example if $C_{1,k} = I$, k=0,...,N then player's 2 controllability subspace is

zero!) it did not really make our problem much easier. An inspection of the

static problem of Section 5.2 shows that if R_1 or R_2 (see (5.2)) is zero

then the static game (5.1)-(5.2) does have a unique solution which is affine

in the information. Similarly for the game (5.34)-(5.37) if the R_{ij}'s or

the $\bar{R}_{ij}$'s are zero. The translation of these conditions to problem (5.51)-

(5.55), assuming that (5.61) holds is that

$$[B_{i,k} \vdots A_k B_{i,k-1} \vdots A_k A_{k-1} B_{i,k-2} \vdots \cdots \vdots A_k A_{k-1} \cdots A_1 B_{i,0}]' Q_{i,k+1} [B_{j,k} \vdots A_k B_{j,k-1} \vdots \cdots$$
$$\cdots \vdots A_k \cdots A_1 B_{j,0}] = 0, \quad i=1, \quad j=2, \quad k=0,1,\ldots,N \qquad (5.62)$$

holds. Then the problem of player 1 is totally independent of $u_{2,k}$, i.e.

it is like setting $R_{ij} = 0$ in (5.38). The meaning of (5.62) is that the

observable, through $Q_{1,k}$, parts of the controllable subspaces of players 1

and 2 are orthogonal in a "stagewise" sense. If (5.61) and (5.62) hold then

player 1 is eventually faced with solving a system of equations like (5.38)

but with $R_{ij} = 0$. This system can be solved uniquely and give a control for

player u_1 linear in the information under invertibility conditions of certain

matrices. (The reason that makes these invertibility assumptions necessary is that the matrices $\bar{A}_{k,\ell}$ arising in the cross terms $u'_{1\ell}\bar{A}_{k\ell}u_{1k}$, $0 \leqslant k \neq \ell \leqslant N$ might make the control problem nonconvex in the $(u_{1,0}, u_{1,1}, \ldots, u_{1,N})$ variable.) Consequently player 2's problem can now be solved, since it becomes a classical LQG problem. For solving player 2's LQG problem no new invertibility assumptions need to be made. Of course if one reverses the roles of i=1, j=2 in (5.62), similar conclusions hold with the roles of players 1 and 2 reversed.

Other cases where the game (5.51)-(5.55) can be solved and give solutions linear in the information can be found by imposing condition (5.61) and additional conditions which guarantee that the assumptions of Theorems 5.6 and 5.7 hold. We will not carry out these procedures here because of the complicated character of the conditions involved.

5.6. Continuous Time Analogues

In this section we will deal with some continuous time analogues of some cases considered for the discrete dynamic stochastic Nash game. Consider the system

$$dx(t) = [A(t)x(t) + B_1(t)u_1(t) + B_2(t)u_2(t)]dt + G(t)dw(t) \qquad (5.63)$$

and the cost functionals

$$J_1 = E[\int_{t_o}^{t_f} [x'(t)Q_1(t)x(t) + u'_1(t)u_1(t) + u'_2(t)R_1(t)u_2(t)]dt] \qquad (5.64)$$

$$J_2 = E[\int_{t_o}^{t_f} [x'(t)Q_2(t)x(t) + u'_2(t)u_2(t) + u'_1(t)R_2(t)u_1(t)]dt]. \qquad (5.65)$$

u_i is chosen by player i as a measurable function of $y_{it} \triangleq \{y_i(s); t_o \leqslant s \leqslant t_o\}$, where

$$dy_i(t) = C_i(t)x(t)dt + R_i dv_i, \quad i = 1,2 \tag{5.66}$$

and u_1, u_2 are such that the solution of (5.63) exists and is unique over $[t_o \ t_f]$. $x(t_o) = x_o$ is a Gaussian random vector with given mean and covariance and dw, dv_1, dv_2 are standard Wiener processes. x_o, dw, dv_1, dv_2 are independent. The initial conditions for y_1, y_2 are zero. $x(t), u_i(t), y_i(t)$ take values in R^n, R^{m_i}, R^{q_i} respectively and the matrices A, B_i, C_i, $Q_i = Q_i' \geqslant 0$, $R_i = R_i' \geqslant 0$, G, $R_i = R_i'$, $(R_i R_i' > 0)$ are piecewise continuous functions of time with appropriate dimensions.

The case in the discrete problem of Section 5.5 where each player uses an affine function of his information, corresponds here to the case where each player uses

$$u_i(t) = \int_{t_o}^{t} [d_s \eta_i(t,s)] y_i(s) + b_i(t) \tag{5.67}$$

where $d_s \eta_i(t,s)$ are deterministic Lebesque-Stieltjes measures and $b_i(t)$ deterministic functions. If player 1 plays u_1 as in (5.67), then under appropriate conditions for player 2's problem a separation principle holds and he will use u_2 of the form (5.67), see [64]. Notice that (5.67) is a compact form of writing the solution of an infinite dimensional filtering problem. The need of infinite dimensional filtering for linear quadratic continuous Nash games was first pointed out in [65]. The study of possible nonlinear solutions to the game of this section is far more difficult than in the discrete time case.

Let us try now to translate condition (5.61) to the continuous time problem. Let $\Phi(t,t_o)$ be the transition matrix which corresponds to $A(t)$.

Lemma 5.2: If

$$C_1(t)\Phi(t,\tau)B_2(\tau) \equiv 0 \qquad t_o \leqslant \tau \leqslant t \leqslant t_f \qquad i \neq j, \qquad (5.68)$$

for $(i,j) = (1,2)$ then the information of player 1 is independent of the control law of player 2.

Proof: From (5.63) and (5.66) using (5.68) we obtain

$$y_1(t) = \int_{t_o}^{t} C_1(s)\Phi(s,t_o)x_o d_s + \int_{t_o}^{t} [\int_{t_o}^{s} C_1(s)\Phi(s,\tau)B_1(\tau)u_1(y_{1,\tau},\tau)d\tau]ds$$

$$+ \int_{t_o}^{t} [\int_{t_o}^{s} C_1(s)\Phi(s,\tau)dw(\tau)]ds + \int_{t_o}^{t} R_1(s)dv_1(s) \qquad (5.69)$$

and the conclusion follows.   ∎

Obviously (5.68) is the analogue of (5.61) for $i=1, j=2$.

Let us now find the analogue of (5.62). The way we derived (5.62) was by imposing the condition that there are no cross terms $u'_{1,k}\bar{A}u_{2,\ell}$ $0 \leqslant k, \ell \leqslant N$ ($\bar{A}$: some matrix) in J_1. Therefore here we demand that

$$(\int_{t_o}^{t} \Phi(t,\tau)B_1(\tau)u_1(\tau)d\tau)'Q_1(t)(\int_{t_o}^{t} \Phi(t,\tau)B_2(\tau)u_2(\tau)d\tau) = 0 \qquad (5.70)$$

for every t and every u_1, u_2. (5.70) can be written equivalently

$$\int_{t_o}^{t}\int_{t_o}^{t} u'_1(\tau)B'_1(\tau)\Phi'(t,\tau)Q_1(t)\Phi(t,\sigma)B_2(\sigma)u_2(\sigma)d\tau d\sigma = 0$$

and thus we conclude that (5.70) is equivalent to

$$B'_1(\tau)\Phi'(t,\tau)Q_1(t)\Phi(t,\sigma)B_2(\sigma) \equiv 0, \ t_o \leqslant \sigma, \tau \leqslant t \qquad t \in [t_o, t_f] \qquad (5.71)$$

(5.71) is clearly the continuous time analogue of (5.62). We can now state the following theorem.

Theorem 5.8: Assume that (5.68) holds with $(i,j) = (1,2)$ and $(i,j) = (2,1)$ and that (5.71) holds. Then for the Nash game (5.63)-(5.66) player 1 can

find his control by solving a control problem with state $y_1(t)$ obeying

(5.69) and a quadratic cost depending only on u_1 and y_1.

Clearly the quadratic cost mentioned in Theorem 5.8 will be of a more general form than the familiar one, and the control problem of player 1 will be of the category considered in [64]. The results of [64] can be used for solving the control problem of player 1 under the assumptions of Theorem 5.8. The invertibility type conditions for solvability of this problem in the discrete time case (see discussion after (62)) have to be made here too. As soon as this control problem is solved player 2 will be faced with a linear quadratic problem of the form treated in [64]. Thus we conclude that the assumptions of Theorem 5.8 in conjunction with the results of [64], can lead with relative ease to the solution of the problem.

We will close this section with a particularly simple case for which the Nash game can be solved without using the results of [64] on the infinite dimensional stochastic regulator problem. Let us assume that

$$
\begin{aligned}
C_2(t)\Phi(t,t_o)B_1(t) &\equiv 0 \\
C_1(t)\Phi(t,t_o)B_2(t) &\equiv 0 \qquad t \in [t_o,t_f] \\
C_1(t) &= C_1(t)Q_1^{1/2}(t), \quad Q_1(t_o) = I.
\end{aligned}
\tag{5.72}
$$

Let us also assume that $Q_1^{1/2}(t)$ is continuously differentiable and invertible on $[t_o,t_f]$. Let

$$
z(t) = Q_1^{1/2}(t)x(t).
\tag{5.73}
$$

From (5.63) and (5.72),(5.73) we obtain

$$
z(t) = Q_1^{1/2}(t)\Phi(t,t_o)x_o + \int_{t_o}^{t} Q_1^{1/2}(t)\Phi(t,\tau)B_1(\tau)u_1(\tau)d\tau + \int_{t_o}^{t} Q_1^{1/2}(t)\Phi(t,\tau)G(\tau)d\tau.
\tag{5.74}
$$

From (5.66) and (5.72) we obtain

$$dy_1(t) = C_1(t)x(t)dt + R_1(t)dv_1(t) = C_1(t)Q_1^{1/2}(t)x(t)dt + R_1(t)dv_1(t)$$

$$= C_1(t)z(t)dt + R_1(t)dv_1(t). \tag{5.75}$$

Let

$$\tilde{A}(t) = [Q_1^{1/2}(t)A(t) + \frac{d}{dt}Q_1^{1/2}(t)]Q_1^{-1/2}(t). \tag{5.76}$$

Then player 1's problem can be written

$$dz(t) = (\tilde{A}(t)z(t) + B_1(t)u_1(t)]dt + G(t)dw(t),$$

$$z(t_o) = x_o$$

$$dy_1(t) = C_1(t)z(t)dt + R_1(t)dv_1(t) \tag{5.77}$$

$$\min\int_{t_o}^{t_f} (z'(t)z(t) + u_1'(t)u_1(t))dt$$

where we omitted the term $u_2'(t)R_1(t)u_2(t)$ since the first two of the assumptions (5.72) guarantee that $y_2(t)$ is independent of u_1, (see Lemma 5.2). (5.77) is a classical stochastic linear regulator problem. Player 1 has to use $E[z(t)|y_{1,t}] = \hat{z}(t)$ and his control law is linear. Player 2 will be faced with a classical linear quadratic control problem with state $(x(t), \hat{z}(t))$ and he will have to calculate $E[x(t)|y_{2,t}]$ and $E[z(t)|y_{2,t}]$ $= Q^{1/2}(t)E[E[x(t)|y_{1,t}]|y_{2,t}]$; i.e. player 2 will use a two step filtering procedure. We thus see that conditions (5.72) single out a class of problem with nonnested y_1 and y_2 which admits a unique solution linear in the estimates $E[x(t)|y_{1,t}]$, $E[x(t)|y_{2,t}]$, $E[E[x(t)|y_{1,t}]|y_{2,t}]$ and there is no need for infinite dimensional filtering.

CHAPTER 6

CONCLUSIONS

The work presented in this thesis answers certain questions in the area of Leader-Follower and Nash differential games. The main topics include existence of closed-loop Nash strategies, necessary conditions for closed-loop Leader-Follower strategies, sufficient conditions for Leader-Follower and Nash strategies with memory and stochastic Nash games with state information. In Chapter 2 we gave conditions which guarantee the existence of linear closed loop Nash strategies. We also established, in this chapter, the uniqueness of analytic closed loop Nash strategies for analytic differential games. In Chapter 3 we gave necessary conditions for closed-loop Leader-Follower strategies. It was shown that the leader's problem is a singular control problem, which explains the peculiar behavior of certain previously studied examples. It was also established that the leader can in general restrict himself to strategies affine in the state and that there are other types of strategies to which the leader can restrict himself without deteriorating his cost. These different types of strategies although guaranteeing the same optimal costs and trajectories, might result in different sensitivity or stability properties. It was also shown in this chapter that the Principle of Optimality holds for Leader-Follower differential games, if and only if it is in the players' common interest to consider the leader's problem as a team problem. In Chapter 4 we dealt with sufficient conditions for Leader-Follower and Nash differential games where the players have recall of previous values of the trajectory. In particular for the Leader-Follower game we considered the case where the Leader's problem is treated as a team

problem by both players. The conditions of this chapter were given in the form of integral equations. In Chapter 5 we considered stochastic Nash games. Results concerning the uniqueness of linear strategies were derived for the static case and translated to the dynamic one. For the dynamic case it was shown that if the control of one player does not affect the information of the other and if at least one player's cost is not affected by the other through the penalization of the state, then under invertibility conditions for certain matrices, the linear solution is the unique one.

There is a plethora of problems to be studied in the area of differential games with state information, besides those studied in this thesis. We will single out some of them arising naturally from the study presented here. Chapter 2 suggests the following areas of research: 1) Uniqueness (or nonuniqueness) of closed loop Nash strategies. 2) Geometric conditions for solvability of the coupled Riccati equations. 3) Efficient algorithms for solving the coupled Riccati equations. Chapter 3 suggests the following areas of research: 1) Generalize the analysis in Chapter 3 to nonstandard control problems where $x_t = \{x(\theta) \mid 0 \leqslant \theta \leqslant t\}$ plays the role of $x(t)$. 2) Generalize the analysis in Chapter 3 to L-F games where the leader uses $u(x_t, t)$. 3) Efficient algorithms for solving the L-F game of Chapter 3. Chapter 4 suggests the following areas of research: 1) Existence and uniqueness of the solutions presented in Chapter 4. 2) Single out simple classes of L-F and Nash games with memory and provide computational procedures. Chapter 5 suggests the following areas of research: 1) In view of the possible nonlinear solutions to the games of this chapter, provide simple sufficient conditions to check optimality of nonlinear solutions. 2) Geometric interpretations of the

matrix-type conditions of this chapter. Of course in all chapters, questions of sensitivity and stability character of the solutions arise especially when the solutions of the games are not unique.

APPENDIX A

PROOF OF PROPOSITION A.1

In [4], Proposition 1 states that given (2.1),(2.2) where $R_{11}, R_{22} > 0$, then if K_1, K_2 satisfying (2.4) exist and $\tilde{A}$ (as in (2.5)) is A.S., then the strategies (2.7) satisfy (2.3). This is not true as the counterexample $\dot{x} = x + u + v$, $-J_2 = J_1 = \int_o^{+\infty} (x^2 + u^2 - 2v^2)dt$ demonstrates. [This example is used in [9] to show that in the zero sum case the linear solution for the game over a finite period of time $[0,T]$ does not, as $T \to +\infty$, tend to the linear solution of the infinite time case.] But if one makes additional assumptions then the conclusion holds.

The correct form of Proposition 1 in [4] is

Proposition A.1: Given the system (2.1) and the two functionals (2.2) where Q_i, R_{ij} are real symmetric matrices and $R_{11}, R_{22} > 0$, if there exist real symmetric matrices K_1, K_2 satisfying (2.4) and (2.5) and either (i) or (ii) hold:

(i) $\quad Q_i + K_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} K_j \geq 0$, $\quad i,j = 1,2$, $\quad i \neq j$.

(ii) The two control problems

$$\dot{x} = (A - B_j R_{jj}^{-1} B_j' K_j)x + B_i u_i$$

$$\min \int_o^{+\infty} (x'[Q_i + K_j R_{jj}^{-1} R_{ij} R_{jj}^{-1} K_j]x + u_i' R_{ii} u_i)dt, \quad i \neq j, \quad i,j = 1,2$$

satisfy the conditions of Theorem 2, p. 167 in [14], then the strategies (2.7) satisfy (2.3) and J_1^*, J_2^* are finite.

APPENDIX B

NONFINITE J_i^*'s

The case where at least one of J_1^*, J_2^* is $\pm\infty$ could also be examined. For example, if we are interested in a linear Nash equilibrium where $J_1^* = \pm\infty$ and J_2^* is finite, then this amounts to seeking $u_i = -L_i^* x$, $i = 1, 2$ where (i), (ii), (iii) hold:

(i) The control problem

$$\dot{x} = (A - B_1 L_1^*) x + B_2 u_2$$

$$\bar{J}_2 = \min \int_0^{+\infty} (x'(Q_2 + L_1^{*'} R_{21} L_1^*) x + u_2 R_{22} u_2) dt$$

has $\bar{J}_2 \ (= J_2^*)$ finite. (For example assume controllability and $Q_2 + L_1^{*'} R_{21} L_1^* \geq 0$).

(ii) The problem

$$\dot{x} = (A - B_2 L_2^*) x + B_1 u_1$$

$$\bar{J}_1 = \int_0^{+\infty} (x'(Q_1 + L_2^* R_{12} L_2^*) x + u_1' R_{11} u_1) dt$$

has $\bar{J}_1 = +\infty$ for every $u_1 = -L_1 x$ (which means roughly that some uncontrollable mode λ of the pair $(A - B_2 L_2^*, B_1)$ which does not lie in the null space of $Q_1 + L_2^* R_{12} L_2^*$ has Re $[\lambda] \geq 0$).

(iii)

$$L_2^* = R_{22}^{-1} B_2' K_2$$

$$K_2(A - B_1 L_1^*) + (A - B_1 L_1^*)' K_2 + (Q_2 + L_1^{*'} R_{21} L_1^*) - K_2 B_2 R_{22}^{-1} B_2' K_2 = 0.$$

Similarly one can form conditions for the cases $J_1^* = J_2^* = \pm\infty$, $J_1^* = -\infty$ and J_2^* finite.

APPENDIX C

PROOF OF 2.32

Consider the matrix differential equation

$$\dot{K}_1 = K_1 A + A'K_1 + Q_1 - K_1 S_1 K_1 - K_1 S_2 K_2 - K_2 S_2 K_1 + K_2 S_{o1} K_2 \tag{C.1}$$

$$\dot{K}_2 = K_2 A + A'K_2 + Q_2 - K_2 S_2 K_2 - K_2 S_1 K_1 - K_1 S_1 K_2 + K_1 S_{o2} K_1 \tag{C.2}$$

where K_1, K_2 are time varying, $t \geq 0$ and $K_1(0) = \Gamma_1$, $K_2(0) = \Gamma_2$ are the initial conditions. Then it follows that for $t \geq 0$ sufficiently small it holds

$$K_1(t) = e^{At}\Gamma_1 e^{A't} + \int_o^t e^{A\sigma}[Q_1 - K_1 S_1 K_1 - K_1 S_2 K_2 - K_2 S_2 K_1 + K_2 S_{o1} K_2] \cdot e^{A'\sigma} d\sigma$$

(and a similar one for $K_2(t)$). The constant matrices Γ_1, Γ_2 solve (2.67), (2.68) if and only if

$$\Gamma_1 = e^{At}\Gamma_1 e^{A't} + \int_o^t e^{A\sigma}[Q_1 - \Gamma_1 S_1 \Gamma_1 - \Gamma_1 S_2 \Gamma_2 - \Gamma_2 S_2 \Gamma_1 + \Gamma_2 S_{o1} \Gamma_2] e^{A'\sigma} d\sigma$$

(and a similar one for Γ_2). Because A is A.S. and Γ_1, Γ_2 constant, the integral

$$I_\infty = \int_o^{+\infty} e^{A\sigma}[Q_1 - \Gamma_1 S_1 \Gamma_1 - \Gamma_1 S_2 \Gamma_2 - \Gamma_2 S_2 \Gamma_1 + \Gamma_2 S_{o1} \Gamma_2] e^{A'\sigma} dx$$

exists. Also

$$\int_t^{+\infty} e^{A\sigma}[Q_1 - \Gamma_1 S_1 \Gamma_1 - \Gamma_1 S_2 \Gamma_2 - \Gamma_2 S_2 \Gamma_1 + \Gamma_2 S_{o1} \Gamma_2] e^{A'\sigma} d\sigma =$$
$$(w = \sigma - t)$$

$$= \int_o^{+\infty} e^{A(w+t)}[Q_1 - \Gamma_1 S_1 \Gamma_1 - \Gamma_1 S_2 \Gamma_2 - \Gamma_2 S_2 \Gamma_1 + \Gamma_2 S_{o1} \Gamma_2] e^{A'(w+t)} dw =$$

$$= e^{At} I_\infty e^{A't}$$

and thus

$$I_\infty = \int_0^t + \int_t^{+\infty} = \int_0^t + e^{At} I_\infty e^{A't}$$

from which we conclude that $I_\infty = \Gamma_1$. A similar result holds for Γ_2. Introducing scaling $\Gamma_1 = \alpha \bar{\Gamma}_2$, $\alpha > 0$ and setting $K_i = \Gamma_i$, we have (2.32).

APPENDIX D

PROOF OF 2.36

It is easy to see that $\|e^{Ft}\| \le \|T\| \; \|T^{-1}\| \; \|e^{\Lambda t}\|$. Let $t > 0$ and

$$
\Lambda = \begin{bmatrix} J_1 & & & \\ & J_2 & & 0 \\ & & \ddots & \\ & 0 & & J_k \end{bmatrix}
$$

where J_i's are the Jordan blocks of dimension $m_1,\ldots,m_k$ ($m_1 + \ldots + m_k = n$).

Let

$$
e^{J_i t} = e^{\lambda_i t} \Delta_i = e^{\lambda_i t} \begin{bmatrix} 1 & \dfrac{t}{1!} & \dfrac{t^2}{2!} & \cdots & \dfrac{t^{m_i-1}}{(m_i-1)!} \\ & 1 & & & \vdots \\ & & \ddots & & \vdots \\ 0 & & & \ddots & 1 \end{bmatrix}
$$

Then

$$
\|e^{\Lambda t}\| = \max_{i=1,\ldots,k} \|e^{\lambda_i t} \Delta_i\| \le e^{\bar\lambda t} \max_{i=1,\ldots,k} \|\Delta_i\|.
$$

We have

$$
\Delta_i = I + \left[\begin{array}{c|c} 0 & t \begin{bmatrix} 1 & \dfrac{t}{2!} & \cdots & \dfrac{t^{i-2}}{(i-1)!} \\ & 1 & & \vdots \\ & & \ddots & \vdots \\ 0 & & & 1 \end{bmatrix} \\ \hline 0 & 0 \end{array} \right] = I + \left[\begin{array}{c|c} 0 & t\bar\Delta_i \\ \hline 0 & 0 \end{array} \right].
$$

So, if $x = (x_1,\ldots,x_i)' \in R^i$, $\tilde x = (x_2,\ldots,x_i)'$

$$\|\Delta_i\| = \sup_{\|x\|=1} \|\Delta_i x\| \leq \sup_{\|x\|=1} \{\|x\| + t\|\bar{\Delta}_i \tilde{x}\|\} =$$

$$= 1 + t \sup_{x_2^2 + \ldots + x_i^2 = 1} \|\bar{\Delta}_i \tilde{x}\| = 1 + t \sup_{\|\tilde{x}\| \leq 1} \|\bar{\Delta}_i \tilde{x}\| = 1 + t\|\bar{\Delta}_i\|.$$

For $\bar{\Delta}_i$ we have

$$\bar{\Delta}_i = \begin{bmatrix} 1 & \frac{t}{2!} & \frac{t^2}{2 \cdot 3} & \cdots & \frac{t^{i-2}}{2 \ldots (i-1)} \\ & \cdot & & & \\ & & \cdot & & \vdots \\ & & & \cdot & \\ 0 & & & & 1 \end{bmatrix} =$$

$$= I + \left[\begin{array}{c|c} & \frac{t}{2} \begin{bmatrix} 1 & \frac{t}{3} & \cdots & \frac{t^{i-3}}{3 \ldots (i-1)} \\ & \cdot & & \\ & & \cdot & \\ 0 & & & 1 \end{bmatrix} \\ \hline 0 & 0 \end{array} \right] = I + \left[\begin{array}{c|c} 0 & \frac{t}{2} \cdot \bar{\bar{\Delta}}_i \\ \hline 0 & 0 \end{array} \right].$$

We have again

$$\|\bar{\Delta}_i\| \leq 1 + \frac{t}{2} \|\bar{\bar{\Delta}}_i\|$$

and thus

$$\|\Delta_i\| \leq 1 + t(1 + \frac{t}{2} \|\bar{\bar{\Delta}}_i\|).$$

Continuing similarly we end up with

$$\|\Delta_i\| \le 1 + t(1 + \frac{t}{2}(1 + \frac{t}{3}(\ldots((1 + \frac{t}{i-2})(1 + \frac{t}{i-1}))\ldots))) =$$

$$= 1 + \frac{t}{1!} + \frac{t^2}{2!} + \ldots + \frac{t^{i-1}}{(i-1)!} \ .$$

Therefore, if $m = \max(m_1, \ldots, m_k)$

$$\|e^{Ft}\| \le \|T\|\|T^{-1}\|\|e^{\Lambda t}\| \le \|T\|\|T^{-1}\|e^{\bar{\lambda}t}(1 + \frac{t}{1!} + \ldots + \frac{t^{m-1}}{(m-1)!}) = \rho \sum_{j=0}^{m-1} (\frac{t^j}{j!}) \ .$$

Direct calculation gives (recall $\bar{\lambda} < 0$)

$$\int_0^{+\infty} \|e^{Ft}\|^2 dt \le \rho^2 \int_0^{+\infty} e^{2\bar{\lambda}t}[\sum_{j=0}^{m-1} (\frac{t^j}{j!})]^2 dt = \rho^2 \int_0^{+\infty} \sum_{i,j=0}^{m-1} e^{2\bar{\lambda}t} \cdot \frac{t^{i+j}}{i!j!} \, dt =$$

$$= \rho^2 \sum_{i,j=0}^{m-1} \frac{(i+j)!}{i!j!} \left(\frac{1}{-2\bar{\lambda}}\right)^{i+j+1} = \rho^2 \pi(-1/\bar{\lambda}) \ . \tag{D.1}$$

APPENDIX E

SUFFICIENT CONDITIONS FOR ASSUMPTION (A)

In this Appendix we give certain conditions under which Assumption (A) (Section 3.2) holds.

<u>Lemma E.1</u>: Let U_ℓ be a subset of U (see (3.1)), defined as

$$U_\ell = \{u \in U | u(x,t) = C(t)x + D(t), \text{ where the } m_1 \times n \text{ matrix } C(\ )$$

and the $m_1 \times 1$ vector $D(t)$ are piecewise continuous $\qquad$ (E.1)

functions of time over $[t_o, t_f]\}$.

Then it holds:

$$\inf_{u \in U_\ell, \ v \in Tu} J_1(u,v) \geq \inf_{u \in U, \ v \in Tu} J_1(u,v) \geq \inf_{u \in U, \ v \in T'u} J_1(u,v) = \inf_{u \in U_\ell, \ v \in T'u} J_1(u,v). \tag{E.2}$$

<u>Proof</u>: The inequalities follow from the facts $U_\ell \subseteq U$, $Tu \subseteq T'u \ \forall \ u \in U$. The last equality is obvious in the light of (3.26) and the proof of Theorem 3.2.

$\Box$

An immediate conclusion of Lemma E.1 is that if

$$\inf_{u \in U_\ell, \ v \in Tu} J_1(u,v) = \inf_{u \in U_\ell, \ v \in T'u} J_1(u,v) \tag{E.3}$$

holds, then Assumption (A) holds (with $U_N^* = U$). For (E.3) to hold, it suffices that the first order necessary conditions for the follower's problem are also sufficient, for each fixed $u \in U_\ell$. More specifically, for fixed $C(t)$, $D(t)$ as in definition (E.1), we consider the problem

$$\text{minimize } h(x(t_f)) + \int_{t_o}^{t_f} M(x, C(t)x + D(t), v, t)dt$$

(E.4)

subject to: $v \in V$

$$\dot{x} = f(x, C(t)x + D(t), v, t), \quad x(t_o) = x_o, \quad t \in [t_o, t_f]$$

and seek conditions under which the first order necessary conditions for an optimal v^* for problem (E.4) (see (3.10-2)-(3.10-4)) are also sufficient. Such conditions can be found in Chapter 5-2 of [39]. We formalize this discussion in the following Proposition.

<u>Proposition E.1</u>: If for each $u \in U_\ell$, the first order necessary conditions (3.10-2)-(3.10-4) for problem (E.4) are also sufficient, then Assumption (A) holds.

The discussion in the present Appendix generalizes clearly to the case where each u^i depends on $h^i(x, t)$ instead of x and to the case where different U_ℓ's are considered; see for example Proposition 3.1(ii).

As an example where Proposition E.1 can be applied, we consider the linear quadratic game of Section 3. Then, Theorem 5, p. 341 of [39] in conjunction with Proposition E.1 yield that if $Q_2 \gtrsim 0$, $R_{22} > 0$, $R_{21} \gtrsim 0$, $K_{2f} \gtrsim 0$ then Assumption (A) holds.

APPENDIX F

PROOF OF THEOREM 3.1

<u>Proof of Theorem 3.1</u>: Let $g \equiv 0$ w.l.o.g. (see [35]). Consider a function $\varphi \in U$, $\varphi = (\varphi^1, \ldots, \varphi^m)$ which has the same continuity and differentiability properties as u^*. Such a φ will be called admissible. Using the known theorems on the dependence of solutions of differential equations on parameters, we conclude that for $\epsilon \in R$, ϵ sufficiently small, $u^* + \epsilon \varphi$ gives rise to a trajectory $\{(x(\epsilon,t),t) | t \in [t_o, t_f]\}$, $x(0,t) = x^*(t)$, and that $x(\epsilon,t)$ is in C^1 w.r. to ϵ. Direct calculation yields

$$\frac{d}{dt} \left(\frac{\partial x(\epsilon,t)}{\partial \epsilon} \right) = [f_x + (u_x + \epsilon \varphi_x) f_u + \sum_{i=1}^m (u_{xx}^i + \epsilon \varphi_{xx}^i) f_i]' \cdot \frac{\partial x(\epsilon,t)}{\partial \epsilon}$$

$$+ f_u' \varphi + \sum_{i=1}^m f_i' \nabla_x h^i \varphi_{y^i}^i, \qquad \frac{\partial x(\epsilon,t)}{\partial \epsilon} \bigg|_{t=t_o} = 0. \tag{F.1}$$

We set

$$z(t) = \frac{\partial x(\epsilon,t)}{\partial \epsilon} \bigg|_{\epsilon=0} \tag{F.2}$$

$$A(t) = f_x + u_x f_u + \sum_{i=1}^m [\nabla_x h^i u_{y^i}^i \nabla_y \nabla_x h^{i'} + \sum_{j=1}^{q_i} u_j^i \nabla_{xx} h_j^i f_i] p \tag{F.3}$$

$$B_1(t) = f_u' \tag{F.4}$$

$$B_2^i(t) = f_i' \nabla_x h^i, \qquad i = 1, \ldots, m \tag{F.5}$$

where A, B_1, B_2^i are evaluated at t, x^*, u^*, u_x^* and, thus, for $\epsilon = 0$, (F.1) can be written as

$$\dot{z} = Az + B_1 \varphi + \sum_{i=1}^m B_2^i \varphi_{y^i}^i, \qquad z(t_o) = 0. \tag{F.6}$$

For fixed φ we consider

$$\bar{J}(\epsilon) = J(u + \epsilon \varphi).$$

Since $\overline{J}(\epsilon)$ is in C^1 w.r. to ϵ and u^* is a local optimum, it must hold

$$\frac{d\overline{J}(\epsilon)}{d\epsilon}\bigg|_{\epsilon=0} = 0.$$

Direct calculation yields

$$\frac{d\overline{J}(\epsilon)}{d\epsilon} = \int_{t_o}^{t_f}\{[L_x + (u_x+\epsilon\varphi_x)L_u + \sum_{i=1}^{m}(u_{xx}^i+\epsilon\varphi_{xx}^i)L_i]'\,\frac{\partial x(\epsilon,t)}{\partial\epsilon}$$

$$+L_u'\varphi + \sum_{i=1}^{m}L_i'\nabla_x h^i\varphi_{y^i}^i\}dt \tag{F.7}$$

Setting

$$\Gamma(t) = L_x + u_x L_u + \sum_{i=1}^{m}[\nabla_x h^i u_{y^i y^i}^i\nabla_x h^{i'} + \sum_{j=1}^{q_i}u_j^i\nabla_{xx}h_j^i]L_i \tag{F.8}$$

$$\Delta_1(t) = L_u' \tag{F.9}$$

$$\Delta_2^i(t) = L_i'\nabla_x h^i, \quad i=1,\ldots,m. \tag{F.10}$$

with Γ, Δ_1, Δ_2^i evaluated at x^*, u^*, u_x^*, we conclude from (F.7)-(F.10) that

$$\int_{t_o}^{t_f}[\Gamma z + \Delta_1\varphi + \sum_{i=1}^{m}\Delta_2^i\varphi_{y^i}^i]dt = 0. \tag{F.11}$$

Therefore (F.11) must hold for every admissible φ. Let $\Phi(t,\tau)$ be the transition matrix of $A(t)$. Let also $\overline{\varphi}(t)$ denote the vector $(\varphi^1(h^1(x^*(t),t),t),\ldots,$ $\varphi^m(h^m(x^*(t),t),t))'$ and $\overline{\varphi}^i(t)$ the vector $\frac{\partial\varphi^i(h^i(x^*(t),t),t)}{\partial x}$. Then from (F.6) we obtain

$$z(t) = \int_{t_o}^{t}\Phi(t,\tau)[B_1(\tau)\overline{\varphi}(\tau) + \sum_{i=1}^{m}B_2^i(\tau)\overline{\varphi}^i(\tau)]d\tau \tag{F.12}$$

$$t\in[t_o,t_f]$$

and substituting in (F.11) we obtain

$$\int_{t_o}^{t_f} \{\Gamma(t)\int_{t_o}^{t} \Phi(t,\tau)[B_1(\tau)\overline{\varphi}(\tau) + \sum_{i=1}^{m}B_2^i(\tau)\overline{\varphi}^i(\tau)]d\tau + \Delta_1(t)\overline{\varphi}(t)$$

$$+ \sum_{i=1}^{m}\Delta_2^i(t)\overline{\varphi}^i(t)\}dt = 0. \tag{F.13}$$

Let $X_{[a,b]}$ denote the indicator function of $[a,b] \subseteq [t_o,t_f]$. We can inter-change the order of integration in (F.13) since the integrated quantities are bounded on $[t_o,t_f] \times [t_o,t_f]$ (Fubini's Theorem). Using the fact $X(c)_{[t_o,b]} = X(b)_{[c,t_f]}$ we have successively

$$\int_{t_o}^{t_f} \int_{t_o}^{t_f} [\Gamma(t)\Phi(t,\tau)B_1(\tau)\overline{\varphi}(\tau) + \Gamma(t)\Phi(t,\tau)\sum_{i=1}^{m}B_2^i(\tau)\overline{\varphi}^i(\tau)] \cdot$$

$$\cdot X(t)_{[\tau,t_f]} d\tau dt = \int_{t_o}^{t_f} [\int_{\tau}^{t_f} \Gamma(t)\Phi(t,\tau)dt]B_1(\tau)\overline{\varphi}(\tau) d\tau +$$

$$+ \sum_{i=1}^{m}\int_{t_o}^{t_f} [\int_{\tau}^{t_f} \Gamma(t)\Phi(t,\tau)dt]B_2^i(\tau)\overline{\varphi}^i(\tau)d\tau. \tag{F.14}$$

By introducing

$$p'(\tau) = \int_{\tau}^{t_f} \Gamma(t)\Phi(t,\tau)d\tau \tag{F.15}$$

(F.13) can be written as

$$\int_{t_o}^{t_f} [p'(\tau)B_1(\tau) + \Delta_1(\tau)]\overline{\varphi}(\tau) + \sum_{i=1}^{m}\int_{t_o}^{t_f} [p'(\tau)B_2^i(\tau) + \Delta_2^i(\tau)] \cdot$$

$$\cdot \overline{\varphi}^i(\tau) d\tau = 0. \tag{F.16}$$

Applying Lemma 3.1 to (F.16), we obtain

$$p'(\tau)B_1(\tau) + \Delta_1(\tau) \equiv 0, \text{ on } [t_o,t_f] \tag{F.17}$$

$$p'(\tau)B_2^i(\tau) + \Delta_2^i(\tau) \equiv 0 \quad \text{on } [t_o, t_f]. \tag{F.18}$$

Using (F.4), (F.5) and (F.9), (F.10) in (F.17), (F.18) we have equivalently (3.20) and (3.21). Differentiation of (F.15) and use of (F.3) and (F.8) give the equivalent to (F.15)

$$-\dot{p} = L_x + f_x p + \sum_{i=1}^{m} \sum_{j=1}^{q_i} u_j^i \nabla_{xx} h_j^i (L_i + f_i p)$$

$$p(t_f) = 0.$$

The assumption $g \equiv 0$, is removed in the known way, resulting in (3.22). $\quad \square$

APPENDIX G

PROOF OF THEOREM 4.1

<u>Proof of Theorem 4.1</u>: Consider the functions

$$H_1 : R^n \times C_n \times L_{\infty,m} \rightarrow C_n$$

$$H_2 : R^n \times C_n \times L_{\infty,m} \rightarrow L_{1,k}$$

$$H_3 : R^n \times C_n \times L_{\infty,m} \rightarrow R^n \tag{G.1}$$

$$J : R^n \times C_n \times L_{\infty,m} \rightarrow R$$

defined for $(\xi,x,u) \in R^n \times C_n \times L_{\infty,m}$ by

$$H_1(\xi,x,u)(t) = x(t) - \int_{t_0}^{t} A(\tau)x(\tau)d\tau - \int_{t_0}^{t} B(\tau)u(\tau)d\tau - x_0$$

$$H_2(\xi,x,u)(t) = \int_{t_0}^{t_f} [d_s\eta(t,s)]x(s) + \int_{t_0}^{t_f} [d_s\eta_1(t,s)]u(s) - q(t) \tag{G.2}$$

$$H_3(\xi,x,u) = \xi - x_0 - \int_{t_0}^{t_f} A(\tau)x(\tau)d\tau - \int_{t_0}^{t_f} B(\tau)u(\tau)d\tau$$

$$J(\xi,x,u) = \frac{1}{2} [\xi'F\xi + \int_{t_0}^{t_f} (x'(t)Q(t)x(t) + u'(t)R(t)u(t))dt]$$

Clearly, H_1, H_3, and J are well defined. To show that H_2 is well defined

it suffices to show that if $u \in L_{\infty,m}$ then $\int_{t_0}^{t_f} [d_s\eta_1(t,s)]u(s) \in L_{1,k}$.

Let $u \in L_{\infty,m}$, $\|u\|_{L_\infty} = M$. Then there exists a

sequence $\{u_n\}_{n=1}^{\infty}$ of continuous functions $u_n:[t_0,t_f] \rightarrow R^m. \ni: u_n(t) \rightarrow u(t)$

a.e. and $|u_n(t)| \leq M + 1 \ \forall \ t \in [t_0,t_f], \forall n$, (see Theorem 3, page 106 in

[49]. $y_n(t) = \int_{t_o}^{t_f} [d_s \eta_1(t,s)] u_n(s)$ is measurable, $|y_n(t)| \leq (M+1) m_1(t)$

and thus $y_n \in L_{1,m}$. Since $u_n \to u$ a.e., by Egoroff's Theorem we have that[1]
$\forall \; \epsilon > 0$, $\mu_\ell(A_n^c) \to 0$ as $n \to +\infty$ where $A_n^c = \{s : s \in [t_o, t_f], |u_n(s) - u(s)| \geq \epsilon\}$.
It holds

$$|y_n(t) - \int_{t_o}^{t_f} [d_s \eta_1(t,s)] u(s)| = |\int_{t_o}^{t_f} [d_s \eta_1(t,s)] (u_n(s) - u(s))| \leq |\int_{A_n} | + |\int_{A_n^c}|$$

$$\leq \epsilon \cdot c_1(t) + (2M+1) c_1(t) \mu_\ell(A_n^c).$$

Since c_1 is finite a.e., letting $n \to +\infty$ we obtain $|\lim y_n(t) - \int_{t_o}^{t_f} [d_s \eta_1(t,s)] u(s)|$
$\leq \epsilon \cdot c_1(t)$, a.e. in $t \in [t_o, t_f]$ where $\lim y_n(t)$ stands for either limsup or
liminf. Since this inequality holds $\forall \epsilon > 0$ we conclude that

$$\int_{t_o}^{t_f} [d_s \eta_1(t,s)] u(s) = \lim y_n(t) \qquad \text{a.e. in } [t_o, t_f]. \tag{G.3}$$

Since $|y_n(t)| \leq (M+1) c_1(t)$ and (A 3) holds, we conclude by Lebesgue's theorem

that
$$\int_{t_o}^{t_f} [d_s \eta_1(t,s)] u(s) \in L_{1,k}.$$

Problem (P) can be written equivalently

$$\text{minimize } J(\xi, x, u)$$
$$\text{subject to } H_i(\xi, x, u) = 0, \quad i = 1, 2, 3 \tag{G.4}$$
$$(\xi, x, u) \in R^n \times C_n \times L_{\infty,m} = \Omega.$$

By Theorem 1, page 220 in [50] we conclude that a sufficient condition for
(ξ^*, x^*, u^*) to solve (G.4) is the existence of a $(\mu, \lambda, k) \in (C_n^*, L_{1,k}, R^n)^*$ such that

---

[1] μ_ℓ denotes Lebesgue measure on $[t_o, t_f]$.

$$J(\xi^*, x^*, u^*) + \langle H_1(\xi^*, x^*, u^*), \mu \rangle + \langle H_2(\xi^*, x^*, u^*), \lambda \rangle + \langle H_3(\xi^*, x^*, u^*), k \rangle$$

$$\leq J(\omega) + \langle H_1(\omega), \mu \rangle + \langle H_2(\omega), \lambda \rangle + \langle H_3(\omega), k \rangle \qquad \forall \omega \in \cap. \tag{G.5}$$

Since the function $\tilde{J}(\omega) = J(\omega) + \langle H_1(\omega), \mu \rangle + \langle H_2(\omega), \lambda \rangle + \langle H_3(\omega), k \rangle$ is convex and Frechet differentiable a necessary and sufficient condition for (G.5) to hold is that

$$d\tilde{J}(\xi^*, x^*, u^*; \zeta, h, v) = 0$$

$$\forall (\zeta, h, v) \in R^n \times C_n \times L_{\infty, m} \tag{G.6}$$

where $d\tilde{J}$ denotes the Frechet differential. Straightforward calculations result in the following explicit form for (G.6).

$$(\xi' F + k') \zeta = 0 \qquad \forall \xi \in R^n \tag{G.7}$$

$$\int_{t_o}^{t_f} x'(t) Q(t) h(t) dt + \int_{t_o}^{t_f} [d\mu'(t)] h(t) + \int_{t_o}^{t_f} \lambda'(t) (\int_{t_o}^{t_f} [d_s \eta(t,s)] h(s)) dt$$

$$+ \int_{t_o}^{t_f} \mu'(t) A(t) h(t) dt - k' \int_{t_o}^{t_f} A(t) h(t) dt = 0 \qquad \forall h \in C_n \tag{G.8}$$

$$\int_{t_o}^{t_f} u'(t) R(t) v(t) dt + \int_{t_o}^{t_f} \mu'(t) B(t) v(t) dt + \int_{t_o}^{t_f} \lambda'(t) (\int_{t_o}^{t_f} [d_s \eta_1(t,s)] v(s)) dt$$

$$- k' \int_{t_o}^{t_f} B(t) v(t) dt = 0, \qquad \forall v \in L_{\infty, m}. \tag{G.9}$$

Use of the unsymmetric Fubini theorem in [47] yields

$$\int_{t_o}^{t_f} \lambda'(t) (\int_{t_o}^{t_f} [d_s \eta(t,s) h(s)) dt = \int_{t_o}^{t_f} [d_s (\int_{t_o}^{t_f} \lambda'(t) \eta(t,s) dt)] h(s) \tag{G.10}$$

$$\int_{t_o}^{t_f} \lambda'(t) (\int_{t_o}^{t_f} [d_s \eta_1(t,s)] v(s)) dt = \int_{t_o}^{t_f} [ds (\int_{t_o}^{t_f} \lambda'(t) \eta_1(t,s) dt)] v(s). \tag{G.11}$$

Using (G.10),(G.11) in (G.8),(G.9) we obtain the sufficiency conditions (4.41), (4.42) wherein we substituted μ by $\mu-k$, and k by $-F\xi^* = -Fx(t_f)$. $\qquad \square$

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